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Superselection Structure and Statistics in Three-Dimensional Local Quantum Theory

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Abstract. We analyze the structure of superselection sectors and the statistics of "charged" fields in general local quantum theories on three-dimensional Minkowski space. We find that physical, charged fields may exhibit braid statistics, including non-abelian braid statistics. We argue that models with (non-abelian) braid statistics can be constructed from (non-abelian) three-dimensional gauge- and matter fields with a topological Chern-Simons term in the effective gauge field action. It is conceivable that our analysis explains some qualitative aspects of the fractional quantum Hall effect and of certain high T_c superconducting materials, but that remains speculative.

1. Introduction

Recently, interest in quantum field theory in three space-time dimensions, in particular in gauge theory with a Chern-Simons term in the Lagrangian, has been revived through the analysis of two-dimensional phenomena in condensed matter physics: In Laughlin's approach to the fractional quantum Hall effect excitations carrying fractional charge and fractional spin, whose statistics is intermediate between Bose- and Fermi statistics, play an important role [1,2]. Excitations with similar properties are also expected to arise in various models of high T_c superconductivity [3]. The detailed mechanisms that give rise to such excitations in strongly correlated, two-dimensional many-body systems do not appear to be well understood, yet. There are, however, phenomenological Landau-Ginzburg type models [1] which exhibit fractionally charged excitations with intermediate statistics which may yield a good description of the excitations in fractional quantum Hall systems. The phenomenological models are abelian gauge theories with a Chern-Simons term in the action; in particular, abelian Higgs models with Abrikosov vortices [4] (originally introduced in the theory of type II superconductors) have been discussed in detail [5]. The effect of the Chern-Simons term is that a region in two-dimensional space through which a magnetic flux threads automatically carries an electric charge proportional to the magnetic flux. Conversely, an electrically charged particle in an abelian gauge theory with a Chern-Simons term automatically carries magnetic flux proportional to its charge. This phenomenon is already encountered at the classical level. The intermediate statistics of such particles can then be understood to be a consequence of the Aharonov-Bohm effect.

It is natural to ask what happens in a non-abelian gauge theory with Chern-Simons term? "Effective" non-abelian gauge fields may be dynamical degrees of freedom of strongly correlated two-dimensional many-body systems. One might wonder whether such systems exhibit particle-like excitations whose statistics is described by some non-abelian version of the Aharonov-Bohm effect. This question is briefly addressed in Sect. 2 of this paper: We argue that the answer is affirmative, although

in this paper we do not present a complete analysis.

In the absence of a detailed understanding of the dynamics of concrete physical systems it may be wise to ask whether the existence of particle-like excitations with fractional spin and intermediate statistics, so-called braid statistics, is compatible with the general principles of local quantum theory? In particular, we are interested in knowing whether particles with non-abelian braid statistics may arise in local quantum theory? To give an answer to such questions is the main purpose of this paper. Our analysis is carried out within the algebraic approach to quantum field theory, as developed by Haag and Kastler [6], Doplicher, Haag and Roberts [7] and extended by Buchholz and Fredenhagen and by Doplicher and Roberts [8]. Although the starting point of our analysis is chosen in accordance with local quantum field theory, neither strict locality nor Poincaré covariance of the theory are fundamental in the derivation of our results. We therefore believe that the main findings presented in this paper extend to non-relativistic many-body systems (including two-dimensional quantum-mechanical lattice systems) with rather minor changes.

The organization of our paper is as follows. In Sect. 2, we briefly recall the basic properties of three-dimensional abelian Higgs models with Chern-Simons term which describe particles with fractional spin and intermediate statistics, so-called anyons [9]. We also present a (somewhat conjectural) description of the main features of non-abelian gauge theories with a Chern-Simons term in three space-time dimensions.

The purpose of the discussion in Sect. 2 is to extract from the discussion of models some basic structural properties of local quantum theory in three space-time dimensions which may serve as a starting point of a model-independent, general analysis.

In Sect. 3, some special features of relativistic quantum physics in three space-time dimensions are recalled. The algebraic approach to local quantum theory is then reviewed, and a basic result, due to Buchholz and Fredenhagen [8], concerning the localization properties of one-particle states (or -representations) is stated.

In Sect. 4, we formulate the starting point of our analysis. We describe the basic properties, in particular the localization properties, of "charge-transfer" operators which play a fundamental role in the theory. This requires some discussion of the duality postulate [7] in the algebraic approach to local quantum theory which expresses the physical idea that the vacuum cannot carry a non-abelian charge [7].

In Sect. 5, we show how one can construct "many-particle sectors" and charged "fields" which make transitions between different sectors. As shown in [8], charged fields can always be localized in space-like cones of arbitrarily small opening angle. The charged fields correspond to the gauge-invariant Mandelstam (string-) operators in gauge theory.

In Sect. 6, we study the statistics of charged fields: It turns out that their statistics is described by two unitary statistics operators, generally distinct, which are shown, in Sect. 7, to determine unitary representations of the braid groups, B_n, on n strands (the groupoids of coloured braids, respectively), for $n = 2, 3, 4, \ldots$.

In Sect. 8, translation- and rotation-covariant sectors are investigated. The spin of an irreducible covariant sector is defined, and a spin addition rule (spin-statistics connection), in agreement with what was previously found in anyon models [5], is established.

Finally, in Sect. 9, we briefly outline the construction of collision (scattering) theory within our general framework, drawing on results in [7,8]. Wave functions of asymptotic states are introduced, and the braid statistics of fields is shown to correspond to a braid statistics of asymptotic particles.

In a companion paper, we shall show that the statistics of charged fields can be described neatly in terms of a family of braid- and fusion matrices with properties identical to those used to describe the monodromy properties of conformal blocks in two-dimensional conformal field theory. These braid- and fusion matrices can be viewed as invariants of three-dimensional quantum field theory and can be used to construct invariants for knots and links in S^3 related to the Jones polynomial.

In many ways, this paper is a review paper. We do, however, believe that some of our observations and results are new.

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2. Three-dimensional gauge theories with braid statistics.

In this section, we summarize some results obtained in the analysis of anyon models, [5,9,10,11]. We consider a three-dimensional U(1)-Higgs model with a (topological) Chern-Simons term

$$\frac{i\mu}{2\pi} \int A \wedge dA \qquad (2.1)$$

in the action which breaks parity invariance. If the gauge coupling constant and the parameters of the Higgs potential of the model are chosen appropriately the physical Hilbert space of the theory contains massive, stable one-particle states carrying magnetic flux (vorticity) +1 or -1 and an electric charge 2μ. They correspond to vortex solutions of the classical equations of motion. The superselection sectors of the theory are labelled by their total vorticity $n \in \mathbf{Z}$. These results can be proven rigorously in a lattice approximation [5].

An application of the Buchholz-Fredenhagen theory [8] to the present situation permits one to construct "field operators" $\psi(C)$, localized in space-like cones, C, with the property that $\psi(C)$ has non-vanishing matrix elements between the vacuum, Ω, of the theory and one-particle states of non-zero vorticity. If the coupling constant μ in front of the Chern-Simons term (2.1) is non-zero the gauge field of the theory is massive [12], and vortices carry an electric charge, q, with

$$q = 2\mu n, \qquad (2.2)$$

n is their vorticity. Particles with magnetic flux and electric charge have been called anyons [9]. The spin, s, of an anyon can be determined by performing a 2π-rotation of a one-anyon state, and one finds that

$$s = \frac{1}{4\mu} q^2 = \mu \ (mod\ \mathbf{Z}). \qquad (2.3)$$

In (2.2) and (2.3), our units are such that the gauge coupling constant, i.e. the electric charge, is equal to one.

Similarly, a physical state with total vorticity $n \in \mathbf{Z}$ has spin (or angular momentum)

$$L_n = n^2 \mu = ns + \frac{n(n-1)}{2} (2\theta) \ (mod\ \mathbf{Z}), \qquad (2.4)$$

with $\theta = \mu = s$ [5]. The number θ will turn out to describe anyon statistics.

Following [7,8] one can develop a Haag-Ruelle collision theory for anyons, provided the mass, m, of the one-anyon states is positive [5]. The momentum-space wave functions,

$$f_n = f_n(p_1, q_1, \cdots, p_n, q_n), \qquad (2.5)$$

describing n asymptotic anyons with 3-momenta $p_1, \cdots, p_n$ on the mass shell $V_m = \{p : p^2 = m^2, p_0 > 0\}$ and charges $q_i = \pm 2\mu$, $i = 1, \ldots, n$, belong to an asymptotic Hilbert space of multi-valued functions: By performing a 2π-rotation on the asymptotic state described by f_n we find that

$$U(2\pi) f_n = exp[2\pi i (\frac{1}{4\mu} (\sum_{j=1}^{n} q_j)^2 - ns)] \, f_n \,. \qquad (2.6)$$

Thus, for $\mu \notin \frac{1}{2}\mathbf{Z}$, f_n cannot be single-valued on momentum space $(V_m)^{\times n}$. Defining

$$M_n = (V_m)^{\times n} \setminus D, \qquad (2.7)$$

where D is the diagonal set $\{(p_1, \ldots, p_n)$: $p_i = p_j$, for some $i \neq j\}$, one can show that f_n determines a single-valued function on the universal cover, $\tilde{M}_n$, of M_n. The fundamental group of M_n is the pure braid group on n strands, P_n.

If the n anyons all have the same charge, q, then the Hilbert space of asymptotic n-anyon-states carries an abelian representation, R, of the braid group on n strands, B_n, given by

$$R(\tau_j^{\pm 1}) = e^{\pm 2\pi i \mu \epsilon^2}, \qquad (2.8)$$

where $\epsilon = q/_{2\mu} = \pm 1$, and τ_i is the i^{th} generator of B_n; see [5,10] and Sect. 7 for further details. Thus if μ is an integer then f_n obeys ordinary Bose statistics, while if μ is half-integer f_n obeys Fermi statistics. For $\mu \notin \frac{1}{2}\mathbf{Z}$, f_n has intermediate ($\theta-$)statistics determined by the statistics parameter $\theta = \mu$, and the following spin-statistics connection holds; see equ. (2.3).

$$s = \theta \qquad (2.9)$$

It is important to describe the observable (gauge-invariant) fields of the model. Among such fields are Wilson loops

$$W(\mathcal{L}) = N(exp\ i \oint_{\mathcal{L}} A_\mu(\xi) d\,\xi^\mu), \tag{2.10}$$

the square of the Higgs field

$$N(\phi^2(x)), \tag{2.11}$$

and gauge-invariant combinations of the Higgs- and the gauge field (Mandelstam operators)

$$N(\phi(x)^*\ exp(i \int_{\gamma_{xy}} A_\mu(\xi) d\xi^\mu)\ \phi(y)). \tag{2.12}$$

In equations (2.10)-(2.12), the symbol N denotes a normal ordering prescription, $\mathcal{L}$ is a (space-like) loop in configuration space, and γ_{xy} is a space-like path from x to y. In an abelian theory the Wilson loop- and Mandelstam operators can be regularized by replacing

$$\int_{\gamma_{xy}} A_\mu(\xi) d\xi^\mu \text{ by } \int d^3\xi\ j^\mu_{xy}(\xi) A_\mu(\xi)$$

where j is a vector-valued function on M^3 with $div\ j_\mu = \delta_x - \delta_y$. In particular, the Wilson loop operators, $W(\mathcal{L})$, can be approximated by bounded operators ("Wilson operators")

$$W(j) = exp\ i \int d^3\xi\ j^\mu(\xi) A_\mu(\xi) \tag{2.13}$$

where j is a vector-valued function with support in some ring surrounding $\mathcal{L}$ located in a spacelike surface, see Fig. 1, and satisfies the equation $div\ j = 0$ which ensures the gauge-invariance of $W(j)$.

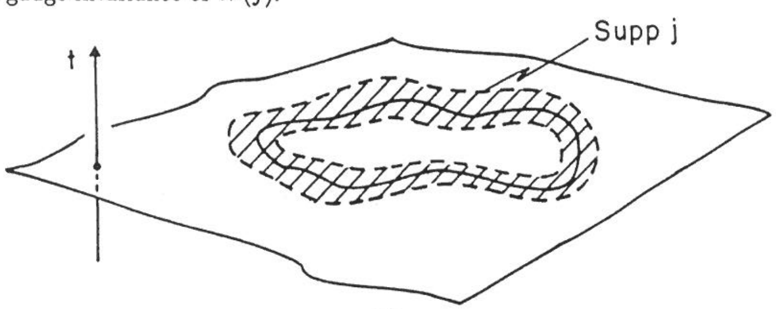

Fig. 1

Similarly, after smearing out $N(\phi^2(x))$ with some test function depending on x and the regularized Mandelstam operators with test functions depending on x and y, these operators can be approximated by bounded operators localized in bounded regions of M^3.

The observable algebra $\mathcal{A}(\mathcal{O})$ associated with a bounded, open region $\mathcal{O} \subset \mathsf{M}^3$ is the (weakly closed) algebra generated by bounded approximations to the operators introduced above localized in $\mathcal{O}$. In this way we obtain a net of (weakly closed) observable algebras, $\{\mathcal{A}(\mathcal{O})\}$, indexed by bounded open regions $\mathcal{O} \subset \mathsf{M}^3$, with $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, and $[A, B] = 0$, for all $A \in \mathcal{A}(\mathcal{O}_1)$ and all $B \in \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated. The algebra of quasi-local observables, $\mathcal{A}$, is defined by

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O}\ bounded} \mathcal{A}(\mathcal{O})}^n \tag{2.14}$$

where $\overline{(\cdot)}^n$ indicates closure in the operator norm. Similarly, for an arbitrary unbounded region $\mathcal{S} \subset \mathsf{M}^3$, e.g. a cone, we define

$$\mathcal{A}(\mathcal{S}) = \overline{\bigcup_{\substack{\mathcal{O}\ bounded \\ \mathcal{O} \subseteq \mathcal{S}}} \mathcal{A}(\mathcal{O})}^n . \tag{2.15}$$

The commutant of $\mathcal{A}(\mathcal{S})$ on a given superselection sector of the theory is denoted by $\mathcal{A}(\mathcal{S})'$. It is automatically weakly closed, but its structure depends on the superselection sector on which it is constructed. In contrast, the algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}$ are sector-independent.

The physical property that the vacuum does not carry a non-abelian charge is expressed by Haag's duality postulate, stating that on the vacuum sector of the theory

$$\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})', \tag{2.16}$$

for any bounded open double cone $\mathcal{O} \subset \mathsf{M}^3$, with $\mathcal{O}'$ the set of points in M^3 space-like separated from $\mathcal{O}$. Equ. (2.16) can be expected to hold in our models, (for our definitions of $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\mathcal{O}')$). In the algebraic approach to local quantum theory, the theory is formulated in terms of the net $\{\mathcal{A}(\mathcal{O})\}$ satisfying (2.16) and the algebras $\mathcal{A}(\mathcal{S})$, $\mathcal{A}$, [6,7,8].

Next, we consider a physical state of arbitrary, but fixed total vorticity n describing, among other particles, a charged vortex (anyon) localized in a space-like cone C_1 containing the origin, 0; (see Sect. 3, Theorem 3.2, for a precise definition of space-like cones). Let C_2 be the space-like cone obtained by rotating C_1 through an angle θ. Let $C_n(\theta)$ denote the operation of transporting the anyon initially localized in C_1 to C_2 along a sequence of cones $C(\theta')$, with $0 \le \theta' \le \theta$, where $C(\theta')$ is obtained from C_1 by rotation through an angle θ'; see Fig. 2.

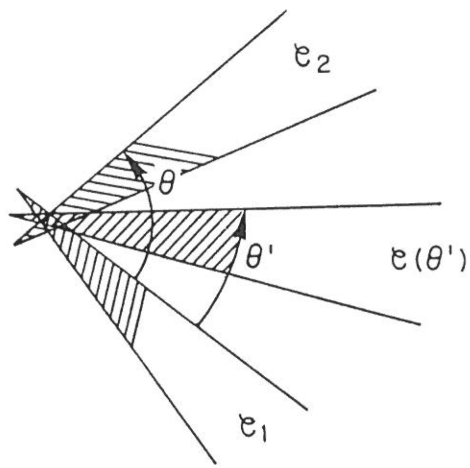

Fig. 2

On the basis of the results in [5], one can argue that $C_n(\theta)$ can be approximated, in the weak operator topology of the sector with total vorticity n, by Wilson operators, $W(j_R)$, localized in the regions $\mathcal{O}_R(\theta)$ indicated in Fig. 3.

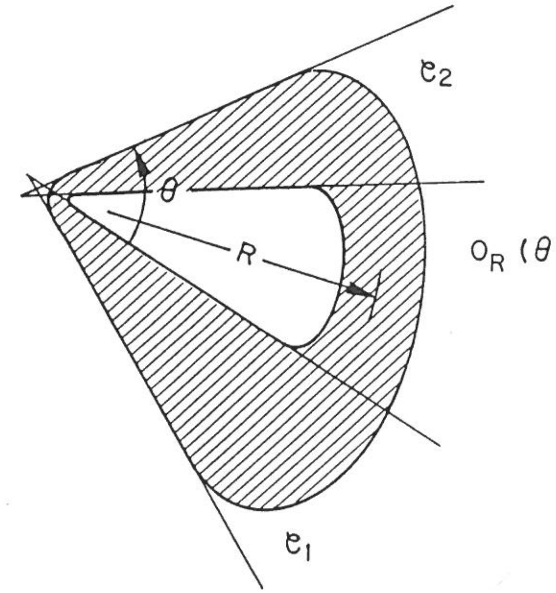

Fig. 3

Explicitly,

$$C_n(\theta) \;=\; \underset{R\to\infty}{w-lim}\; W(j_R), \tag{2.17}$$

for a suitable choice of the sequence $\{j_R\}$, $R = 1, 2, 3, \dots$, of currents.

Let $\mathcal{S}$ be a space-like cone (or the causal complement of a space-like cone) containing $\bigcup_{0\le\theta'\le\theta} C(\theta')$. From our choice of the sequence $\{j_R\}$, with $supp\ j_R \subseteq \mathcal{O}_R(\theta)$, see Fig. 3, and from (2.17) it follows that

$$C_n(\theta) \;\in\; \mathcal{A}(\mathcal{S})_n^-, \tag{2.18}$$

where $\mathcal{A}(\mathcal{S})_n^-$ denotes the closure of the algebra $\mathcal{A}(\mathcal{S})$ in the weak operator topology of the sector with total vorticity n.

Let $\mathcal{S}_{12} = \{x \in \mathrm{M}^3 : (x-y)^2 < 0, \forall\, y \in C_1 \cup C_2\}$ denote the causal complement of the space-like cones C_1 and C_2. It follows easily from the definition (2.15) of $\mathcal{A}(\mathcal{S}_{12})$ and from (2.17) that $C_n(\theta)$ commutes with all operators in $\mathcal{A}(\mathcal{S}_{12})$, i.e.

$$C_n(\theta) \;\in\; \mathcal{A}(\mathcal{S}_{12})'_n,$$

where $\mathcal{A}(\mathcal{S})'_n$ denotes the commutant of the algebra $\mathcal{A}(\mathcal{S})$ on the sector with total vorticity n. Hence

$$C_n(\theta) \in \mathcal{A}(\mathcal{S}_{12})'_n \cap \mathcal{A}(\mathcal{S})^-_n . \qquad (2.19)$$

Next, let $C_n(\theta - 2\pi)$ denote the operation of moving the vortex initially localized in C_1 to C_2 along a sequence of cones $C(\theta')$, where $C(\theta')$ is a rotation of C_1 through an angle θ', with $2\pi - \theta \leq \theta' \leq 0$; see Fig. 4.

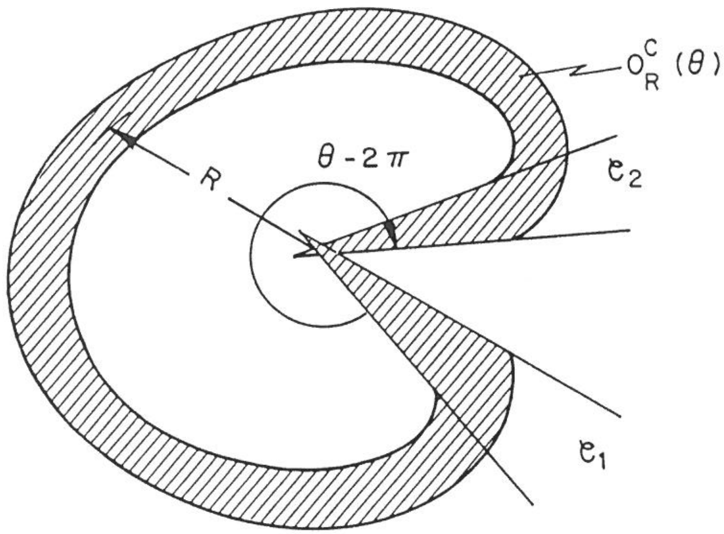

Fig. 4

It is interesting and important to ask whether the operators $C_n(\theta)$ and $C_n(\theta - 2\pi)$ are identical, or not? A priori there is no reason why $C_n(\theta)$ and $C_n(\theta - 2\pi)$ should be identical, since in three space-time dimensions the path of rotations through increasing angles $\theta' \in [0, \theta]$ and the path of rotations through decreasing angles $\theta' \in [\theta - 2\pi, 0]$ are not homotopic, as paths in the three-dimensional Poincaré group. Their composition (with the orientation of one path reversed) is a non-contractible loop in the Poincaré group $\mathcal{P}_+^\uparrow = i\,S\,O(2,1)$. The analysis of the anyon models in [5] shows that

$$C_n(\theta) = e^{i\varphi_n} C_n(\theta - 2\pi), \qquad (2.20)$$

where the phase $\varphi_n = (1 - 2n)\mu$, (see (2.4)), depends on the vorticity n of the sector on which the anyon is transported from C_1 to C_2. If the algebra $\mathcal{A}$ of all quasi-local observables is represented irreducibly on the sector labelled by n then $C_n(\theta)^* C_n(\theta - 2\pi)$ must be a multiple of the identity, by Schur's lemma. Hence the relation (2.20) is a very general fact not specific of the model we consider.

It follows from interesting results of Buchholz and Fredenhagen [8] that one can choose $C_n(\theta)$ or $C_n(\theta - 2\pi)$ to be sector-independent, i.e. independent of n. Equ. (2.20) then shows that it is impossible to choose $C_n(\theta)$ and $C_n(\theta - 2\pi)$ to be sector-independent. It should be emphasized that the phase factor $e^{i\varphi_n} = C_n(\theta)\, C_n(\theta - 2\pi)^*$ only depends on the dynamics of the theory and the sector on which anyon transport is carried out.

Next, let C_1, C_2, C_3 and C_4 be four space-like cones such that $C_1 \cup C_2$ and $C_3 \cup C_4$ are space-like separated, as indicated in Fig. 5.

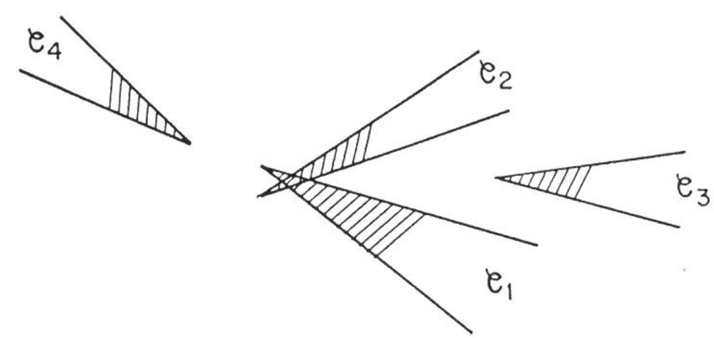

Fig. 5

Let $C_n(1,2)$ denote some anyon transport from C_1 to C_2 and $C_n(3,4)$ an anyon transport from C_3 to C_4. One may ask whether the two operators $C_n(1,2)$ and $C_n(3,4)$ commute, (as one might guess from the circumstance that the regions $C_1 \cup C_2$ and $C_3 \cup C_4$ are space-like separated). Actually, using (2.17) and the "dual" commutation relations between Wilson operators and "vortex creation operators" one finds that

$$C_n(1,2)\, C_n(3,4) = e^{i\varphi}\, C_n(3,4)\, C_n(1,2),$$

where φ is a phase factor proportional to $\pm\,\mu$. One can check that if φ vanished anyons would have ordinary permutation- (Bose- or Fermi-) statistics, rather than braid statistics, in conflict with the results of [5].

The upshot of our discussion is that anyon transport, or, more generally, "charge" transport, in three-dimensional gauge theories (with a Chern-Simons term in the effective gauge field action) can be path- and sector-dependent and that this is intimately related to the braid statistics of anyons. In Sect. 4 we shall describe a starting point - see assumptions (C1)-(C3) - for a general, model-independent analysis which incorporates the insights just gained in a reasonably intuitive way.

Our analysis will leave open the possibility that "charged" fields in local quantum theory might exhibit non-abelian braid statistics, i.e. a statistics described by higher-dimensional, non-abelian representations of the braid groups. In the abelian theories discussed in [5,9,10] only one-dimensional, abelian representations of the braid groups appear. One must therefore ask whether there are gauge theories with a Chern-Simons term,

$$i\frac{k}{4\pi}\int tr\,(A\wedge dA+\frac{2}{3}A\wedge A\wedge A),\ k\in\mathbb{Z}, \tag{2.21}$$

in the action, besides non-topological terms involving the gauge field A and some charged matter fields, ψ, exhibiting non-abelian braid statistics? Although such theories have not been analyzed in detail (see, however, [12,13]), it is a plausible conjecture that they describe "coloured" particles with non-abelian braid statistics.

More precisely, consider a gauge theory with a non-abelian, simply connected compact gauge group, G, and action

$$\begin{aligned} S[A] &= i\frac{k}{4\pi}\int tr(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)+\frac{1}{g^2}\int tr(F^2) \\ &+ \lambda\int\bar{\psi}(\not{D}_A+m)\psi\,. \end{aligned} \tag{2.22}$$

One can argue that this theory does not confine (colour-) charge, for $k\neq 0$, and that the statistics of physical charged fields or -particles is independent of the values of g ($0<g\leq\infty$), $\lambda>0$ and $m>0$, i.e. that it is an invariant of quantum field theory only depending on k and on the nature and number of the matter fields $\psi,\bar{\psi}$.

Recently, the purely topological Chern-Simons theory with action given by (2.21) has been solved exactly by Witten [13]. He finds that the "statistics" of static colour sources is described by certain non-abelian representations of the braid groups which he uses to propose a novel approach to the construction of invariants for links and knots generalizing the famous Jones polynomial. One can argue that Chern-Simons theory is the large-scale asymptote of non-topological theories whose dynamics is given by (2.22). Hence the statistics of physical, charged fields in such theories may be expected to be described by the non-abelian representations of the braid groups that Witten finds. This will be elaborated upon, elsewhere.

3. Local, Relativistic Quantum Theory in Three Space-Time Dimensions

According to Wigner [14], a relativistic particle is described by a unitary, irreducible representation of the quantum mechanical Poincaré group, $\tilde{\mathcal{P}}_+^\uparrow$, the universal covering group of the proper Poincaré group. In three space-time dimensions

$$\mathcal{P}_+^\uparrow = S0(2,1) \times \mathbb{R}^3 .$$

The homogeneous Lorentz group $S0(2,1)$ is isomorphic to $SL(2,\mathbb{R})$ which can be pictured as an open, full torus, i.e. it is homeomorphic to $\mathbb{R}^2 \times S^1$. The circle S^1 corresponds to the subgroup of space rotations. Thus $\pi_1(SL(2,\mathbb{R})) = \mathbb{Z}$, and the covering space is homeomorphic to $\mathbb{R}^3$. The topology of $SL(2,\mathbb{R})$ has interesting consequences for the structure of relativistic physics in three dimensions.

Unitary, irreducible representations of $\tilde{\mathcal{P}}_+^\uparrow$ can be constructed by choosing a Lorentz-invariant subset

$$V_m = \{p \in \mathbb{R}^3 : p^2 = m^2\}, \;\; m^2 \in \mathbb{R}, \tag{3.1}$$

of momentum space. For those representations which describe physical particles, m^2 is non-negative; $m \geq 0$ is the mass of the particle. Then V_m has two disconnected components, and one picks the component

$$V_m^+ = \{p \in \mathbb{R}^3 : p^2 = m^2, \;\; p^0 \geq 0\}. \tag{3.2}$$

One then fixes an energy-momentum vector $p \in V_m^+$ and considers the subgroup of $\widetilde{S0}(2,1)$ which leaves p invariant, isomorphic to the little group. Unitary, irreducible representations of $\tilde{\mathcal{P}}_+^\uparrow$ can be constructed from unitary, irreducible representations of the little group corresponding to a mass hyperboloid V_m^+.

For $m > 0$, the little group is the group of space rotations. Its covering group is the additive group of $\mathbb{R}$, so its irreducible representations are labelled by a real number, s, the spin of the particle. The little group for massless particles, with $m = 0$, is also given by $\mathbb{R}$, and its irreducible representations are labelled by a real number, s, the helicity of the particle.

In conclusion, a relativistic particle in three space-time dimensions is characterized by its mass $m \geq 0$ and its "spin" (spin or helicity) $s \in \mathbb{R}$. These results are due to Bargmann [14].

Next, we wish to discuss local, relativistic quantum theories in three space-time dimensions admitting a particle interpretation. We shall see that for certain simple topological reasons connected with three-dimensional Minkowski space there are two essentially different types of theories. The starting point of our analysis is taken over from a basic paper of Buchholz and Fredenhagen [8] which generalizes the fundamental analysis of Doplicher, Haag and Roberts [7]. The analysis of Buchholz and Fredenhagen concerned theories in four or more dimensions. We shall see that, in three dimensions, some essential changes are required and new phenomena appear.

Common to the analyses contained in [7,8] and in this paper is the algebraic formulation of local, relativistic quantum physics proposed by Haag and Kastler [6]: The observables of the theory which can be measured in open, bounded space-time regions $\mathcal{O} \subset \mathbb{M}^3$ are elements of some von Neumann algebra $\mathcal{A}(\mathcal{O})$ with unit. The algebra of all local observables, $\mathcal{A}$, is defined as the C^*-inductive limit of the local algebras, $\mathcal{A}(\mathcal{O})$, i.e.

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathbb{M}^3} \mathcal{A}(\mathcal{O})}, \tag{3.3}$$

where the closure $\overline{}$ is taken in the operator norm. If C is an unbounded region in $\mathbb{M}^3$ we define

$$\mathcal{A}(C) = \overline{\bigcup_{\mathcal{O} \subset C} \mathcal{A}(\mathcal{O})}, \tag{3.4}$$

where, again, the closure is taken in the operator norm. If two space-time regions, C_1 and C_2, are space-like separated we write $C_1 \times C_2$, and locality is formulated as the statement that

$$[A,B] = 0, \quad \text{for } A \in \mathcal{A}(C_1), \;\; B \in \mathcal{A}(C_2). \tag{3.5}$$

Let $\mathcal{A}^c(C)$ denote the subalgebra of all those operators in $\mathcal{A}$ which commute with all operators in $\mathcal{A}(\mathcal{C})$. Then locality implies that, for $\mathcal{C}_1 \times \mathcal{C}_2$,

$$\mathcal{A}(\mathcal{C}_1) \subset \mathcal{A}^c(\mathcal{C}_2). \tag{3.5a}$$

For a space-time region $\mathcal{C}$ we let $\mathcal{C}'$ denote its space-like complement, i.e.

$$\mathcal{C}' = \{x \in \mathsf{M}^3 : (x-y)^2 < 0, \text{ for all } y \in \mathcal{C}\}. \tag{3.6}$$

Then (3.5) implies that

$$\mathcal{A}(\mathcal{C}) \subseteq \mathcal{A}^c(\mathcal{C}'). \tag{3.7}$$

Of course, it is also always assumed that if $\mathcal{C}_1 \subseteq \mathcal{C}_2$ then $\mathcal{A}(\mathcal{C}_1) \subseteq \mathcal{A}(\mathcal{C}_2)$ and that $\mathcal{A}(\mathcal{C}_3 \cup \mathcal{C}_4) \supseteq \mathcal{A}(\mathcal{C}_3) \vee \mathcal{A}(\mathcal{C}_4)$, (the C^*algebra generated by $\mathcal{A}(\mathcal{C}_3)$ and $\mathcal{A}(\mathcal{C}_4)$).

Next, we formulate relativistic covariance. Let $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ be some Poincaré transformation and let $\mathcal{C}$ be some space-time region. We define

$$\mathcal{C}_{(\Lambda,x)} = \{y \in \mathsf{M}^3 : \Lambda^{-1}(y-x) \in \mathcal{C}\}. \tag{3.8}$$

We assume that $\mathcal{P}_+^\uparrow$ is represented on $\mathcal{A}$ by a group of *automorphisms $\{\alpha_{(\Lambda,x)} : (\Lambda, x) \in \mathcal{P}_+^\uparrow\}$ such that

$$\alpha_{(\Lambda,x)}(\mathcal{A}(\mathcal{C})) \subseteq \mathcal{A}(\mathcal{C}_{(\Lambda,x)}). \tag{3.9}$$

<u>Remark.</u> α is a *automorphism of $\mathcal{A}$ if α is linear, $\alpha(\mathcal{A}) = \mathcal{A}$, $\alpha(A \cdot B) = \alpha(A)\alpha(B)$, for all A, B in $\mathcal{A}$, $\alpha(A^*) = \alpha(A)^*$, for all $A \in \mathcal{A}$. The subgroup of *automorphisms representing space-time translations $(\mathbb{1}, x)$ is denoted by $\{\alpha_x : x \in \mathsf{M}^3\}$, the subgroup of space rotations,

$$\{(\Lambda, 0) : \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix}\} \tag{3.10}$$

by $\{\alpha_\theta : -\pi \le \theta < \pi\}$.

In the next section, we shall add an assumption (Haag duality [7]) expressing the idea that the net $\{\mathcal{A}(\mathcal{O}) : \mathcal{O} \subset \mathsf{M}^3\}$ is maximal, i.e. chosen to contain as many operators as compatible with the requirement of locality, and expressing the assumption that the vacuum does not carry a non-abelian charge.

So far, we have described the structure of the observables of the theory. In order to extract physical information from such a theory, we must study representations, π, of the algebra $\mathcal{A}$ of local observables on (separable) Hilbert spaces, $\mathcal{H}$. In general, a C^*algebra admits a vast number of inequivalent representations most of which are uninteresting for particle physics. To bring order into this situation, we adopt a selection criterion formulated by Borchers (see [8,15]) that singles out those representations of $\mathcal{A}$ which are relevant for particle physics: A representation π of $\mathcal{A}$ is called <u>covariant</u> if there is a strongly continuous unitary representation, $U \equiv U_\pi$, of $\tilde{\mathcal{P}}_+^\uparrow$ on the representation space $\mathcal{H} \equiv \mathcal{H}_\pi$ of π such that

$$\pi(\alpha_{(\Lambda,x)}(A)) = U(\Lambda, x)\pi(A)U(\Lambda, x)^{-1}, \tag{3.11}$$

for all $(\Lambda, x) \in \tilde{\mathcal{P}}_+^\uparrow$ and all $A \in \mathcal{A}$.

<u>Definition 3.1.</u>

(1) A covariant representation π of $\mathcal{A}$ is called a <u>positive-energy representation</u> if the joint spectrum, $\sum$, of the generators, $(P_0, \vec{P})$, of space-time translations $\{U(x) \equiv U(\mathbb{1}, x) : x \in \mathsf{M}^3\}$ satisfies the relativistic spectrum condition, i.e.

$$\sum \subseteq \bar{V}^+ = \{p \in \mathsf{R}^3 : p^2 \ge 0, p_0 \ge 0\} \tag{3.12}$$

(2) A positive-energy representation π of $\mathcal{A}$, on $\mathcal{H}$ is called a <u>(massive) one-particle representation</u> if the set $V_m^+ = \{p : p^2 = m^2, p_0 > 0\}$ is contained in $\sum$ and

$$\sum \subseteq V_m^+ \cup \{p : p^2 \ge M^2, p_0 > 0\} \tag{3.13}$$

for some $M > m > 0$. [The representation, U, of $\tilde{\mathcal{P}}_+^\uparrow$ restricted to the spectral subspace $\mathcal{H}_m \subset \mathcal{H}$ corresponding to the subset V_m^+ of $\sum$ is then a direct sum of the irreducible representations, characterized by their spins $s \in \mathsf{R}$, that were discussed at the beginning of this section; m and s are the mass and spin of a particle.]

(3) A positive-energy representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is called a (massive) vacuum representation if $\sum$ contains 0, and

$$\sum \subseteq \{0\} \cup \{p : p^2 \geq \mu^2, p_0 > 0\}, \tag{3.14}$$

for some $\mu > 0$, called mass gap.

□

In [8], it is shown that **Definition 3.1** still makes sense if one only assumes that space-time translations are unitarily implemented and $\sum \subseteq \bar{V}^+$; i.e. rotations and Lorentz boosts are not important. The reason for this is a result of Borchers and Buchholz [16] which says that locality and the assumption that $\sum \subseteq \bar{V}^+$ already imply that the lower boundary, $\partial_l \sum$, of $\sum$ is Lorentz-invariant, and $\sum \backslash \partial_l \sum$ is Lorentz-invariant. However, in our analysis space rotations will play an important role, and we shall later assume that the entire Poincaré group is represented on $\mathcal{H}$ by unitary operators.

It should be mentioned that, as has been shown by Borchers [15], the operators $U_\pi(x)$ can be chosen to belong to the weak closure, $\overline{\pi(\mathcal{A})}^w$, of $\pi(\mathcal{A})$, i.e. energy- and momentum operators can be approximated by local observables.

We now recall a basic result due to Buchholz and Fredenhagen [8]:

Theorem 3.2. Let $(\pi, \mathcal{H})$ be a massive one-particle representation of $\mathcal{A}$ on a separable Hilbert space[1] $\mathcal{H}$. Then

(1) There exists an irreducible, massive vacuum representation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{A}$ such that, for arbitrary $A \in \mathcal{A}$ and every sequence of points $x \in \mathrm{M}^3$ tending to space-like infinity

$$w - \lim_x \pi(\alpha_x(A)) = \omega_0(A) \cdot \mathbb{1} \tag{3.15}$$

where $\omega_0(A) = \langle \Omega, \pi_0(A)\Omega \rangle$, and $\Omega \in \mathcal{H}_0$ is the unique (Poincaré-invariant) vacuum state.

[1] *$\mathcal{H}$ will usually be separable if it contains a finite number of distinct massive one-particle states.*

(2) Let $\mathcal{O}$ be some open double cone whose closure $\bar{\mathcal{O}}$ is space-like separated from the origin $0 \in \mathrm{M}^3$. Let $a \in \mathrm{M}^3$. Then the region

$$C = a + \bigcup_{\lambda > 0} \lambda \mathcal{O} \tag{3.16}$$

is called a space-like cone with apex a.

Let C be an arbitrary space-like cone. Then the restrictions of π and π_0 to the algebra $\mathcal{A}^c(C)$ are unitarily equivalent, i.e., there exists an isometry V_C of $\mathcal{H}$ onto $\mathcal{H}_0$ such that

$$V_C \pi(A) = \pi_0(A) V_C, \quad \text{for } A \in \mathcal{A}^c(C). \quad \square \tag{3.17}$$

This important result motivates the following definiton.

Definition 3.3. [8] A representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is said to be localizable in cones relative to a vacuum representation $(\pi_0, \mathcal{H}_0)$ if for any space-like cone C there exists an isometry V_C from $\mathcal{H}$ onto $\mathcal{H}_0$ such that

$$V_C \pi(A) = \pi_0(A) V_C, \quad \text{for all } A \in \mathcal{A}^c(C). \tag{3.18}$$

The family of all such representations of $\mathcal{A}$ is denoted by $\mathcal{L}_{\pi_0}$. □

It will turn out that not only massive one-particle representations, but also representations of $\mathcal{A}$ describing many ("charged") particles belong to $\mathcal{L}_{\pi_0}$, [8]. We thus conclude that those representations $(\pi, \mathcal{H}_\pi)$ of $\mathcal{A}$ which are relevant for particle physics, at least in theories with a positive mass gap and only massive, isolated one-particle states,[2] are all vacuum representations and all covariant positive-energy representations $(\pi, \mathcal{H}_\pi)$ of $\mathcal{A}$ which are localizable in cones relative to some vacuum representation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{A}$. The family of all such representations is denoted by $\mathcal{L}_{\pi_0}^{cov}$.

[2] *Representations localizable in cones appear to be adequate in theories with massless particles, as well [17].*

The total Hilbert space, $\mathcal{H}_{phys}$, of the theory is defined as

$$\mathcal{H}_{phys.} = \bigoplus_{\pi_0} \bigoplus_{\pi \in \mathcal{L}^{cov}_{\pi_0}} \mathcal{H}_\pi. \tag{3.19}$$

It carries a strongly continuous unitary representation, U, of $\tilde{\mathcal{P}}^{\uparrow}_{+}$,

$$U = \bigoplus_{\pi_0} \bigoplus_{\pi \in \mathcal{L}^{cov}_{\pi_0}} U_\pi \tag{3.20}$$

satisfying the relativistic spectrum condition. The representation spaces $\mathcal{H}_\pi$ are called (superselection) sectors of the theory.

Our purpose, in the next sections, will be to construct unobservable field operators, ψ, which make transitions between different superselection sectors, $\mathcal{H}_\pi$ and $\mathcal{H}_{\pi'}$, π, π' in $\mathcal{L}_{\pi_0}$, of the theory and to analyze the spin carried by these operators and their commutation relations, i.e., the statistics of the field operators ψ. We shall encounter two very different structures: Local fields carrying integral or half-integral spin and satisfying (para-)Bose or (para-)Fermi statistics [7,8]; and fields localized in space-like cones carrying fractional spin ($\notin \frac{1}{2}\mathbb{Z}$) and satisfying braid statistics. The last case represents the new structure analyzed in this paper; see also [32,18].

There is no loss of generality, if we restrict our analysis to a single vacuum represenation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{A}$ and the associated family $\mathcal{L}_{\pi_0}$ of representations π localizable in cones relative to the given π_0.

Remark. Throughout our analysis it is crucial that we are considering a space-time of dimension larger than two. In two space-time dimensions, a one-particle representation can be connected to two distinct vacuum representations. The analogue of Theorem 3.2, (1), would be

$$w-\lim_{x\to\pm\infty} \pi(\alpha_x(A)) = \omega_0^{\pm}(A)\cdot 1\!\!1,$$

where ω_0^+ and ω_0^- can be distinct vacuum states [19].

Let $\mathcal{O}$ be a double cone. Then, in two-dimensional space-time, $\mathcal{O}'$ is the union, $\mathcal{O}^- \cup \mathcal{O}^+$, of two disjoint wedges, $\mathcal{O}^-$ and $\mathcal{O}^+$, space-like separated from $\mathcal{O}$; $\mathcal{O}^-$ lies to the left and $\mathcal{O}^+$ to the right of $\mathcal{O}$. The correct modification of Theorem 3.2, (2) is then that there exist isometries $V^{\pm}$ from $\mathcal{H}_\pi$ to $\mathcal{H}_0^{\pm}$ such that

$$V^{\pm}\pi(A) = \pi_0(A)V^{\pm}, \text{ for all } A \in \mathcal{A}(\mathcal{O}^{\pm}),$$

if $\mathcal{O}$ is chosen sufficiently large, [8,19].

The ensuing general mathematical structure of local, relativistic quantum theory in two space-time dimensions, in particular, the braid statistics of fields, has been studied in detail in [19,32,22,18,31].

4. Localized Morphisms, Duality and Enlarged Algebras.

Let $\mathcal{A} = \overline{\bigcup_{\mathcal{O} \subset \mathbf{M}^3} \mathcal{A}(\mathcal{O})}$ be as specified in Sect. 3, let $(\pi_0, \mathcal{H}_0)$ be a vacuum representation of $\mathcal{A}$ and $(\pi, \mathcal{H})$ a (covariant, positive-energy) representation localizable in cones relative to $(\pi_0, \mathcal{H}_0)$, see Definition 3.3, i.e. there is an isometry, V_C, from $\mathcal{H}$ to $\mathcal{H}_0$ such that

$$V_C \pi(A) = \pi_0(A) V_C, \text{ for all } A \in \mathcal{A}^c(C), \tag{4.1}$$

where C is an arbitrary space-like cone. It will not matter whether π is a massive one-particle representation or whether π describes massless particles. Massive one-particle representations were only important to motivate the notion of representations localizable in cones; (Theorem 3.2).

Since C in (4.1) can be chosen at ones convenience and V_C is an isometry, it follows that $\|\pi(A)\| = \|\pi_0(A)\|$, for all $A \in \bigcup_{\mathcal{O} \subset \mathbf{M}^3} \mathcal{A}(\mathcal{O})$, hence for all $A \in \mathcal{A}$. Therefore any $\pi \in \mathcal{L}_{\pi_0}$ can be regarded as a representation of $\pi_0(\mathcal{A})$. We shall identify $\pi_0(\mathcal{A})$ and $\mathcal{A}$ in the following.

Thanks to the existence of isometries, V_C, from $\mathcal{H}_\pi$ to $\mathcal{H}_0$, we may define a representation ρ_C of $\mathcal{A}$ on $\mathcal{H}_0$ equivalent to π by setting

$$\rho_C(A) = V_C \pi(A) V_C^{-1}, \text{ for all } A \in \mathcal{A}. \tag{4.2}$$

By (4.1)

$$\rho_C(A) = \pi_0(A) \equiv A, \text{ for all } A \in \mathcal{A}^c(C). \tag{4.3}$$

We shall say that ρ_C <u>is localized in</u> C. Let C_1 and C_2 be arbitrary space-like cones. Then, for a given π, ρ_{C_1} and ρ_{C_2} are unitary equivalent representations of $\mathcal{A}$, by (4.1),(4.2). Hence there exists a unitary operator $\Gamma(C_1, C_2)$ on $\mathcal{H}_0$ such that

$$\Gamma(C_1, C_2)\rho_{C_2}(A) = \rho_{C_1}(A)\,\Gamma(C_1, C_2). \tag{4.4}$$

for all $A \in \mathcal{A}$. If $A \in \mathcal{A}^c(C_1) \cap \mathcal{A}^c(C_2)$ then by (4.1)

$$\Gamma(C_1, C_2)\, A = A\, \Gamma(C_1, C_2). \tag{4.5}$$

Unfortunately, it is not true, in general, that $\rho_C(\mathcal{A}) \subseteq \mathcal{A}$, and it does <u>not</u> follow from (4.5) that $\Gamma(C_1, C_2) \in \mathcal{A}$. Given two representations, ρ_1 and ρ_2, it is therefore not possible, without further assumptions, to define their composition, $\rho_1 \circ \rho_2(A) \equiv \rho_1(\rho_2(A)), A \in \mathcal{A}$. But composition of representations, ρ_i, localized in cones relative to $(\pi_0, \mathcal{H}_0)$ is a crucial device to describe multi-particle states of arbitrary charge and end up with a theory that has a precise mathematical structure. At this point one may envisage two different sets of auxiliary assumptions permitting the composition of representations ρ_i leading to qualitatively different quantum theories.

(A) Doplicher, Haag and Roberts [7] start from the assumption of <u>duality for double cones</u>: Let $\mathcal{B}$ be a C^*- (or von Neumann) algebra of bounded operators acting on a Hilbert space $\mathcal{H}_0$. We define $\mathcal{B}'$ to consist of all those bounded operators on $\mathcal{H}_0$ commuting with all operators in $\mathcal{B}$; $\mathcal{B}'$ is called the commutant of $\mathcal{B}$ and is automatically <u>weakly</u> closed, i.e. a von Neumann algebra. The weak closure of $\mathcal{B}$ is identical to $(\mathcal{B}')' \equiv \mathcal{B}''$. Doplicher, Haag and Roberts assume that, on the vacuum sector $\mathcal{H}_0$,

$$\mathcal{A}(\mathcal{O}')' = \mathcal{A}(\mathcal{O}), \tag{4.6}$$

for any bounded, open double cone $\mathcal{O} \subset \mathbf{M}^3$. [In more precise notation, (4.6) is the statement that $\pi_0(\mathcal{A}(\mathcal{O}'))' = \pi_0(\mathcal{A}(\mathcal{O}))$.] Under suitable assumptions, (4.6) implies that the vacuum sector cannot carry a non-abelian charge [20].

Doplicher, Haag and Roberts then consider theories for which the representations ρ_C are <u>strictly localized</u>:[3] There exists a bounded, open double cone $\mathcal{O}$ such that, with $\rho_C \equiv \rho_\mathcal{O}$,

$$\rho_\mathcal{O}(A) = A, \text{ for all } A \in \mathcal{A}(\mathcal{O}')$$

Then, for $A \in \mathcal{A}(\mathcal{O})$, $\rho_\mathcal{O}(A)$ commutes with all operators in $\mathcal{A}(\mathcal{O}')$, so that, by (4.6), $\rho_\mathcal{O}(A) \in \mathcal{A}(\mathcal{O})$. Hence $\rho_\mathcal{O}(\mathcal{A}) \subseteq \mathcal{A}$, i.e. $\rho_\mathcal{O}$ is a *morphism of $\mathcal{A}$; (ρ is a *morphism of $\mathcal{A}$ if ρ is a linear map from $\mathcal{A}$ into $\mathcal{A}, \rho(A \cdot B) = \rho(A)\rho(B)$, and $\rho(A^*) = \rho(A)^*$).

As shown by Doplicher, Haag and Roberts the unobservable (charged) fields reconstructed from strictly localized morphisms of $\mathcal{A}$ obey (para-)Bose or (para-)

[3] *This is well motivated in theories without gauge fields.*

Fermi statistics and carry integral spin or half-integral spin, respectively. [The spin-statistics connection is well known in three-dimensional, local quantum field theory, in the sense of Wightman [33]. In the algebraic setting, the three-dimensional theories were not considered explicitly in [7], but the results claimed above are implicit in [7].]

(B) Buchholz and Fredenhagen [8] introduce a version of the duality postulate more convenient for their purposes: If $\mathcal{C}$ is an arbitrary space-like cone then

$$\overline{\mathcal{A}^c(\mathcal{C})}^w = \mathcal{A}(\mathcal{C})', \tag{4.7}$$

where $\overline{\mathcal{B}}^w$ denotes the weak closure of an algebra, $\mathcal{B}$, of operators acting on $\mathcal{H}_0$; (again we omit explicit mentioning of the vacuum representation). By (4.5), the unitary intertwiners $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ introduced in equ. (4.4) then belong to $\overline{\mathcal{A}(\mathcal{C})}^w$, where $\mathcal{C}$ is a space-like cone, or the causal complement of a space-like cone, containing $\mathcal{C}_1 \cup \mathcal{C}_2$. Moreover, by (4.3),

$$\rho_{\mathcal{C}}(\mathcal{A}(\tilde{\mathcal{C}})) \subseteq \overline{\mathcal{A}(\tilde{\mathcal{C}})}^w, \tag{4.8}$$

for any cone $\tilde{\mathcal{C}}$ containing $\mathcal{C}$.

The duality postulate (4.7) could serve as an adequate starting point for the general analysis of statistics presented in Sects. 5 - 8. It does not, in general, exclude braid statistics, because it does not imply that $\Gamma(\mathcal{C}_1, \mathcal{C}_2) \in \overline{\mathcal{A}(\mathcal{C}_1)}^w \vee \overline{\mathcal{A}(\mathcal{C}_2)}^w$. However, we shall base our analysis on a somewhat weaker hypothesis, (C), formulated below, which still suffices to develop a fairly precise general theory. The reader will verify without difficulty that hypothesis (C) is a consequence of (B). We feel that our hypothesis (C) reflects more directly the insights gained in Sect. 2 than the somewhat abstract-duality postulate (4.7). [Technically, (B) would, however, simplify the analysis in Sect. 7.]

Definition 4.1. A domain $\mathcal{S} \subseteq \mathbb{M}^3$ is called simple iff $\mathcal{S}$ is a space-like cone, or $\mathcal{S}$ is the causal complement of a space-like cone. Let $\mathcal{S}$ be a simple domain, and let $\mathcal{C}_1$ and $\mathcal{C}_2$ be space-like cones. We define the algebras

$$\mathcal{B}(\mathcal{S}) = \mathcal{A}(\mathcal{S}')', \tag{4.9}$$

$$\mathcal{B}(\mathcal{C}_1 \cup \mathcal{C}_2) = (\mathcal{A}(\mathcal{C}_1') \cap \mathcal{A}(\mathcal{C}_2'))'. \tag{4.10}$$

Note that, in general, $\mathcal{B}(\mathcal{C}_1 \cup \mathcal{C}_2)$ properly contains $\overline{\mathcal{A}(\mathcal{C}_1)}^w \cup \overline{\mathcal{A}(\mathcal{C}_2)}^w$.

It is easily verified that assumption (B) implies the following properties of morphisms and intertwiners.

Proposition 4.2. Let $\rho_{\mathcal{C}_1}$ and $\rho_{\mathcal{C}_2}$ be two equivalent representations of $\mathcal{A}$ on $\mathcal{H}_0$ localized in space-like cones $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and let $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$ be a unitary operator on $\mathcal{H}_0$ intertwining $\rho_{\mathcal{C}_1}$ and $\rho_{\mathcal{C}_2}$. Then

(a) $\rho_{\mathcal{C}_1}(\mathcal{A}(\mathcal{S})) \subseteq \overline{\mathcal{A}(\mathcal{S})}^w$, for any simple domain $\mathcal{S} \supseteq \mathcal{C}_1$;

(b) $\Gamma(\mathcal{C}_1, \mathcal{C}_2) \in \mathcal{B}(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \overline{\mathcal{A}(\mathcal{S})}^w$, for any simple domain $\mathcal{S} \supseteq \mathcal{C}_1 \cup \mathcal{C}_2$. □

(C) The following three properties, motivated by the study of three-dimensional gauge theories, are required henceforth.

(C1) Let ρ be a representation of $\mathcal{A}$ on $\mathcal{H}_0$ localized in a space-like cone $\mathcal{C}$. Then, for any simple domain $\mathcal{S} \supseteq \mathcal{C}$,

$$\rho(\mathcal{A}(\mathcal{S})) \subseteq \overline{\mathcal{A}(\mathcal{S})}^w. \tag{4.11}$$

(C2) If $\rho_{\mathcal{C}_1}$ and $\rho_{\mathcal{C}_2}$ are unitarily equivalent representations of $\mathcal{A}$ on $\mathcal{H}_0$ localized in space-like cones $\mathcal{C}_1, \mathcal{C}_2$, respectively, and if $\mathcal{S}$ is any simple domain containing $\mathcal{C}_1 \cup \mathcal{C}_2$ then there is a unitary intertwiner, $\Gamma(\mathcal{C}_1, \mathcal{C}_2)$, such that

$$\Gamma(\mathcal{C}_1, \mathcal{C}_2)\, \rho_{\mathcal{C}_2}(A) = \rho_{\mathcal{C}_1}(A)\, \Gamma(\mathcal{C}_1, \mathcal{C}_2), \tag{4.12}$$

for all $A \in \mathcal{A}$, and

$$\Gamma(\mathcal{C}_1, \mathcal{C}_2) \in \mathcal{B}(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \overline{\mathcal{A}(\mathcal{S})}^w \tag{4.13}$$

Finally, we require the following property.

(C3) [4] If $\mathcal{C}$ is a simple domain and if $\mathcal{S}$ is a simple domain containing $\mathcal{C}$ such that $\mathcal{S}'$ contains some space-like cone then

$$\mathcal{B}(\mathcal{C}) \cap \overline{\mathcal{A}(\mathcal{S})}^{w} = \overline{\mathcal{A}(\mathcal{C})}^{w}. \tag{4.14}$$

Remarks. (1) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are space-like separated space-like cones then there are two minimal simple domains, $\mathcal{S}_1$ and $\mathcal{S}_2$, containing $\mathcal{C}_1 \cup \mathcal{C}_2$; see Fig. 6.

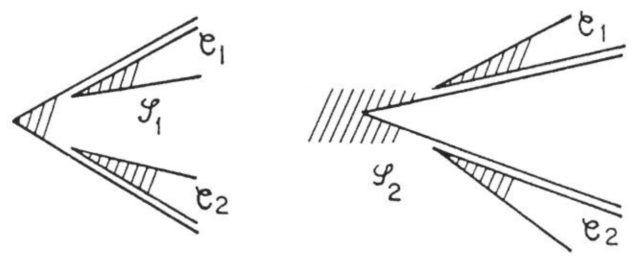

Fig. 6

It then follows from (C2) that there are two intertwiners $\Gamma_1 \equiv \Gamma_1(\mathcal{C}_1, \mathcal{C}_2) \in \overline{\mathcal{A}(\mathcal{S}_1)}^{w}$ and $\Gamma_2 \equiv \Gamma_2(\mathcal{C}_1, \mathcal{C}_2) \in \overline{\mathcal{A}(\mathcal{S}_2)}^{w}$ such that

$$\Gamma_i \rho_{\mathcal{C}_2}(A) = \rho_{\mathcal{C}_1}(A)\Gamma_i, \quad \text{for all } A \in \mathcal{A}, \tag{4.15}$$

$i = 1, 2$.

(2) Suppose that Γ_1 and Γ_2 are arbitrary unitary operators intertwining $\rho_{\mathcal{C}_1}$ and $\rho_{\mathcal{C}_2}$. Then

$$\Gamma_1^* \rho_{\mathcal{C}_1}(A)\Gamma_1 = \Gamma_2^* \rho_{\mathcal{C}_1}(A)\Gamma_2, \quad \text{or}$$

$$\rho_{\mathcal{C}_1}(A) = (\Gamma_2\Gamma_1^*)^* \rho_{\mathcal{C}_1}(A)\Gamma_2\Gamma_1^*, \quad \text{for all } A \in \mathcal{A}.$$

Hence $\Gamma_2\Gamma_1^* \in \rho_{\mathcal{C}_1}(\mathcal{A})'$. The representation $\rho_{\mathcal{C}_1}$ is irreducible iff $\rho_{\mathcal{C}_1}(\mathcal{A})' = \{\lambda \mathbb{I} : \lambda \in \mathbb{C}\}$. Then it follows that

$$\Gamma_2 = e^{i\theta}\Gamma_1, \tag{4.16}$$

[4] *This property is of a technical nature and can be omitted if one limits the scope of the analysis to $\rho_{\mathcal{C}}$'s which are irreducible.*

for some phase $e^{i\theta}$, generally $\neq 1$. But if the commutant of $\rho_{\mathcal{C}_1}(\mathcal{A})$ is non-trivial, i.e., if $\rho_{\mathcal{C}_1}$ is reducible, then intertwiners are unique only up to elements in $\rho_{\mathcal{C}_1}(\mathcal{A})'$, and (4.13) fails for some intertwiners.

The algebras $\{\mathcal{B}(\mathcal{C})\}$ and properties (C1) - (C3), above, define the mathematical structure on which our analysis is based. In the next section, we shall combine (C1) and (C2) with ideas of Buchholz and Fredenhagen, in order to define the composition of representations localized in cones yielding representations of arbitrary "charge" which describe multi-particle states. We shall then reconstruct field operators, ψ, and analyze their statistics. We find, in Sect. 6, that our framework allows for the possibility of braid statistics. In Sects. 5 and 6, we shall need (C3), (or else limit our analysis to irreducible representations $\rho_{\mathcal{C}}$).

5. Extended Algebras, the Composition of Sectors, and the Construction of Field Operators.

Let $\mathcal{C}_a$ be some space-like cone. We define a partial order, $\leq$, on M^3 by setting

$$x \leq y \quad \text{iff} \quad \mathcal{C}_a + x \supseteq \mathcal{C}_a + y.$$

The family of algebras $\{\overline{\mathcal{A}((\mathcal{C}_a + x)')}^w\}$ forms an increasing net of subalgebras of $B(\mathcal{H}_0)$ with respect to $\geq$; (here $B(\mathcal{H}_0)$ is the algebra of all bounded operators on $\mathcal{H}_0$). We define an enlarged, auxiliary algebra, $\mathcal{B}^{\mathcal{C}_a}$, to be the closure in the operator norm of $\bigcup_x \overline{\mathcal{A}((\mathcal{C}_a + x)')}^w$. By definition, $\mathcal{B}^{\mathcal{C}_a}$ only depends on $\{\mathcal{C}_a + x : x \in \mathsf{M}^3\}$, so that $\mathcal{B}^{\mathcal{C}_a+x}$, for all $x \in \mathsf{M}^3$.

Let $\rho_\mathcal{C}$ be a representation of $\mathcal{A}$ localized in $\mathcal{C}$. We recall that by the definition of representations localizable in cones, see Definition 3.1, there exists a representation $\rho_{\mathcal{C}_1}$ equivalent to $\rho_\mathcal{C}$ with $\mathcal{C}_1 \subseteq \mathcal{C}_a + x$. Let $\Gamma(\mathcal{C}, \mathcal{C}_1)$ be a unitary operator intertwining $\rho_\mathcal{C}$ and $\rho_{\mathcal{C}_1}$. By (4.4)

$$\rho_\mathcal{C}(A) = \Gamma(\mathcal{C}, \mathcal{C}_1)\rho_{\mathcal{C}_1}(A)\Gamma(\mathcal{C}, \mathcal{C}_1)^*, \quad \text{for all } A \in \mathcal{A}.$$

For $A \in \mathcal{A}((\mathcal{C}_a + x)')$, $\rho_{\mathcal{C}_1}(A) = A$, hence

$$\rho_\mathcal{C}(A) = \Gamma(\mathcal{C}, \mathcal{C}_1)A\ \Gamma(\mathcal{C}, \mathcal{C}_1)^*. \tag{5.1}$$

Hence $\rho_\mathcal{C}$ is weakly continuous on $\mathcal{A}((\mathcal{C}_a+x)')$ and we may extend $\rho_\mathcal{C}$ to $\overline{\mathcal{A}((\mathcal{C}_a + x)')}^w$ by setting

$$\rho_\mathcal{C}^{\mathcal{C}_a}(B) \equiv \Gamma(\mathcal{C}, \mathcal{C}_1)B\Gamma(\mathcal{C}, \mathcal{C}_1)^*, \tag{5.2}$$

for all $B \in \overline{\mathcal{A}((\mathcal{C}_a + x)')}^w$. This construction can be carried out for all $x \in \mathsf{M}^3$ and defines an extension $\rho_\mathcal{C}^{\mathcal{C}_a}$ of $\rho_\mathcal{C}$ to the algebra $\mathcal{B}^{\mathcal{C}_a}$, defined above.

Proposition 5.1.

(1) Let $\rho_\mathcal{C}$ be a representation of $\mathcal{A}$ on $\mathcal{H}_0$ localized in $\mathcal{C}$. Then there exists a unique *endomorphism, $\rho_\mathcal{C}^{\mathcal{C}_a}$ from $\mathcal{B}^{\mathcal{C}_a}$ into $B(\mathcal{H}_0)$ which is weakly continuous on the algebras $\overline{\mathcal{A}((\mathcal{C}_a + x)')}^w$, for arbitrary $x \in \mathsf{M}^3$, and which coincides with $\rho_\mathcal{C}$ on $\mathcal{A}$.

(2) If $\mathcal{C}_a + x \subseteq \mathcal{C}'$, for some x, then $\rho_\mathcal{C}^{\mathcal{C}_a}$ is a *morphism of the algebra $\mathcal{B}^{\mathcal{C}_a}$, (i.e., $\rho_\mathcal{C}^{\mathcal{C}_a}(\mathcal{B}^{\mathcal{C}_a}) \subseteq \mathcal{B}^{\mathcal{C}_a}$) and an isometry, i.e.,

$$\|\rho_\mathcal{C}^{\mathcal{C}_a}(B)\| = \|B\|, \quad \text{for all } B \in \mathcal{B}^{\mathcal{C}_a}.$$

Proof. Part (1) follows from (5.1), as shown above. In order to prove (2), note that by assumption (C1), (4.11),

$$\rho_\mathcal{C}(\mathcal{A}(\mathcal{C}_a + y)') \subseteq \overline{\mathcal{A}((\mathcal{C}_a + y)')}^w,$$

whenever y is such that $(\mathcal{C}_a + y)' \supseteq \mathcal{C}$. Hence, using (5.2),

$$\rho_\mathcal{C}^{\mathcal{C}_a}(\overline{\mathcal{A}((\mathcal{C}_a + y)')}^w) \subseteq \overline{\mathcal{A}((\mathcal{C}_a + y)')}^w \tag{5.3}$$

By hypothesis, $\mathcal{C}$ is such that $(\mathcal{C}_a + x)' \supseteq \mathcal{C}$, for some x. But then the algebras $\overline{\mathcal{A}((\mathcal{C}_a + y)')}^w$, with y such that $(\mathcal{C}_a + y)' \supseteq \mathcal{C}$, generate $\mathcal{B}^{\mathcal{C}_a}$, and hence (5.3) implies that $\rho_\mathcal{C}^{\mathcal{C}_a}(\mathcal{B}^{\mathcal{C}_a}) \subseteq \mathcal{B}^{\mathcal{C}_a}$. That $\rho_\mathcal{C}^{\mathcal{C}_a}$ is an isometry follows from equ. (5.2). ■

Definition 5.2. The set of representations, $\rho_\mathcal{C}$, of $\mathcal{A}$ on $\mathcal{H}_0$ belonging to $\mathcal{L}_{\pi_0}$ and obeying (C1) and (C2), ((4.11)-(4.13)), which are localized in a given space-like cone $\mathcal{C}$ is denoted by $\mathcal{L}_\mathcal{C}$. Choose an auxiliary cone $\mathcal{C}_a \subseteq \mathcal{C}'$. For ρ_1 and ρ_2 in $\mathcal{L}_\mathcal{C}$, we define their product by setting

$$\rho_1 \circ \rho_2(A) = \rho_1^{\mathcal{C}_a}(\rho_2(A)), \quad \text{for all } A \in \mathcal{A}. \tag{5.4}$$

Note that, since $\mathcal{A} \subset \mathcal{B}^{\mathcal{C}_a}$ and by Proposition 5.1 (2) $\rho_2(\mathcal{A}) \subseteq \mathcal{B}^{\mathcal{C}_a}$, hence $\rho_1^{\mathcal{C}_a}(\rho_2(A))$ is well defined, for all $A \in \mathcal{A}$. We have the following result.

Theorem 5.3. [8]

(1) If ρ_1 and ρ_2 belong to $\mathcal{L}_\mathcal{C}$ then $\rho_1 \circ \rho_2 \in \mathcal{L}_\mathcal{C}$.

(2) $\rho_1 \circ \rho_{2|\mathcal{A}}$ does not depend on the choice of the auxiliary cone $\mathcal{C}_a \subset \mathcal{C}'$.

(3) If $\hat{\rho}_i$ is equivalent to ρ_i, $i = 1, 2$, and $\hat{\rho}_1, \hat{\rho}_2$ belong to $\mathcal{L}_{\hat{\mathcal{C}}}$, for some space-like cone $\hat{\mathcal{C}}$, then $\rho_1 \circ \rho_2$ and $\hat{\rho}_1 \circ \hat{\rho}_2$ are equivalent.

Proof. Using (C1) and (C2), Sect. 4, and Proposition 5.1, one observes that the proof of Theorem 4.2 in [8] applies in the present situation without change. ∎

As a corollary we note that, for $\rho_i \in \mathcal{L}_{C_i}$, with $C_i \subseteq C_a'$, for some auxiliary space-like cone $C_a, i = 1, \cdots, n$, the representation

$$\rho_1^{C_a} \circ \cdots \circ \rho_{n-1}^{C_a} \circ \rho_n$$

is equivalent to some representation $\rho_C \in \mathcal{L}_C$.

Proposition 5.4. Let C_1 and C_2 be space-like separated space-like cones (i.e., $C_1 \times C_2$) such that $C_1' \cap C_2'$ contains an auxiliary cone C_a. Let $\rho_i \in \mathcal{L}_{C_i}$, $i = 1, 2$. Then

$$\rho_1^{C_a} \circ \rho_2^{C_a} = \rho_2^{C_a} \circ \rho_1^{C_a} \text{ on } \mathcal{B}^{C_a}.$$

Proof. Thanks to property (C2), Sect. 4, the proof is identical to the proof of Proposition 4.3 in [8].

Note that it follows from Proposition 5.4 that, for ρ_1 and ρ_2 in $\mathcal{L}_C$, $\rho_1^{C_a} \circ \rho_2$ and $\rho_2^{C_a} \circ \rho_1$ are unitary equivalent.

Let us rephrase our findings in more conventional Hilbert space language. Let

$$\omega_0(A) \equiv < \Omega, A\Omega >, \quad A \in \mathcal{A} \tag{5.5}$$

denote the vacuum state; (Ω is the vacuum in $\mathcal{H}_0$, i.e. the unique (up to a phase) unit vector in $\mathcal{H}_0$ invariant under $U_0(\Lambda, x)$, for all $(\Lambda, x) \in \tilde{\mathcal{P}}_+^\uparrow$). Given an arbitrary $\rho \in \mathcal{L}_C$, we define the state

$$\omega_0 \circ \rho(A) \equiv \omega_0(\rho(A)) \tag{5.6}$$

on $\mathcal{A}$. With an arbitrary state φ on $\mathcal{A}$, (i.e., a positive, linear functional on $\mathcal{A}$ normalized such that $\varphi(\mathbb{1}) = 1$) we can associate a representation π_φ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\varphi$ containing a cyclic vector ξ_φ such that

$$\varphi(A) = < \xi_\varphi, \pi_\varphi(A)\xi_\varphi > . \tag{5.7}$$

This is the so-called Gel'fand-Naimark-Segal construction (analogous to the Wightman reconstruction theorem).

The total Hilbert space $\mathcal{H}_{tot.}$ of the theory can now be defined as

$$\mathcal{H}_{tot.} = \bigoplus_{[\rho]:\rho \in \mathcal{L}_{\pi_0}} \mathcal{H}_{[\rho]} \tag{5.8}$$

where $[\rho]$ denotes the unitary equivalence class represented by ρ and $\mathcal{H}_{[\rho]} \equiv \mathcal{H}_{\omega_0 \circ \rho}$. It carries a representation $\pi_{tot.}$ of $\mathcal{A}$ given by

$$\pi_{tot.} = \bigoplus_{[\rho]:\rho \in \mathcal{L}_{\pi_0}} \pi_{[\rho]}, \tag{5.9}$$

with $\pi_{[\rho]} \equiv \pi_{\omega_0 \circ \rho}$. If ρ is the identity morphism $\mathcal{H}_{[\rho]} = \mathcal{H}_0$, and $\pi_{[\rho]} = \pi_0$. Hence $\mathcal{H}_{tot}$ contains the vacuum sector $\mathcal{H}_0$, and π_{tot} contains the vacuum representation, π_0, as a subrepresentation.

It is conceivable that $\mathcal{L}_{\pi_0}$ contains representations which are not Poincaré-covariant in the sense of equation (3.11), or which are not positive-energy representations, in the sense of Definition 3.1. Later on, we shall specialize to positive-energy representations, but at the present stage of our analysis this is not a relevant concept.

Our purpose is now to construct field operators $\psi_{\rho C}$, localized in space-like cones $C \subset \mathsf{M}^3$, which map a sector $\mathcal{H}_{[\hat{\rho}]}$, $\hat{\rho} \in \mathcal{L}_{\pi_0}$, to the sector $\mathcal{H}_{[\hat{\rho} \circ \rho_C]}$. Our construction proceeds as follows: Given some morphism $\rho \in \mathcal{L}_{\pi_0}$, we construct an isometry, T_ρ, from $\mathcal{H}_0$ to $\mathcal{H}_{[\rho]}$, by setting

$$T_\rho \, \Omega = \xi_\rho, \tag{5.10}$$

where ξ_ρ is the cyclic vector associated with $\omega_0 \circ \rho$ by the Gel'fand-Naimark-Segal construction;

$$T_\rho \, \rho(A)\Omega = \pi_{[\rho]}(A)\xi_\rho, \tag{5.11}$$

for all $A \in \mathcal{A}$. Suppose that $\rho \in \mathcal{L}_C$. Then $\rho(A) = A$, for all $A \in \mathcal{A}(C')$; see (4.3). But by the Reeh-Schlieder theorem [34], Ω is cyclic for $\mathcal{A}(C')$, if C' is a non-empty, open set. Hence the subspace $\{\rho(A)\Omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_0$ if ρ is localized in a space-like cone C. Therefore equ. (5.11) defines T_ρ on a dense subspace of $\mathcal{H}_0$. Next, we observe that, by (5.11),

$$\begin{aligned} < T_\rho\rho(A)\Omega,\ T_\rho\rho(B)\Omega >_{\mathcal{H}_{[\rho]}} &= < \pi_{[\rho]}(A)\xi_\rho, \pi_{[\rho]}(B)\xi_\rho >_{\mathcal{H}_{[\rho]}} \\ &= < \xi_\rho, \pi_{[\rho]}(A^*B)\xi_\rho >_{\mathcal{H}_{[\rho]}} \\ &= \omega_0 \circ \rho(A^*B) \\ &= < \Omega, \rho(A^*B)\Omega >_{\mathcal{H}_0} \\ &= < \rho(A)\Omega,\ \rho(B)\Omega >_{\mathcal{H}_0} . \end{aligned}$$

Thus T_ρ preserves the scalar product and, since it is densely defined, T_ρ extends to an isometry from $\mathcal{H}_0$ into $\mathcal{H}_{[\rho]}$. But by the G.N.S. construction, $\{\pi_{[\rho]}(A)\xi_\rho : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_{[\rho]}$, so the range of T_ρ is $\mathcal{H}_{[\rho]}$. Hence T_ρ is invertible, and T_ρ^{-1} is an isometry from $\mathcal{H}_{[\rho]}$ to $\mathcal{H}_0$.

The operator T_ρ intertwines the representations $(\pi_{[\rho]}, \mathcal{H}_{[\rho]})$ and $(\rho, \mathcal{H}_0)$ of $\mathcal{A}$, i.e.,

$$\pi_{[\rho]}(A)\, T_\rho \;=\; T_\rho\, \rho(A). \tag{5.13}$$

For,

$$\begin{aligned} \pi_{[\rho]}(A)\, T_\rho\, \rho(B)\,\Omega &= \pi_{[\rho]}(A)\, \pi_{[\rho]}(B)\, \xi_\rho \\ = \pi_{[\rho]}(A \cdot B)\xi_\rho &= \pi_{[\rho]}(A \cdot B)\, T_\rho\, \Omega \\ = T_\rho\, \rho(A \cdot B)\,\Omega &= T_\rho\, \rho(A)\rho(B)\,\Omega. \end{aligned}$$

It is easy to extend the definition of T_ρ to an arbitrary sector $\mathcal{H}_{[\hat\rho]}, \hat\rho \in \mathcal{L}_{\pi_0}$, mapping it to $\mathcal{H}_{[\hat\rho\circ\rho]}$, by setting

$$T_\rho \Big|_{\mathcal{H}_{[\hat\rho]}} \;=\; T_{\hat\rho\circ\rho}\, T_{\hat\rho}^{-1} \Big|_{\mathcal{H}_{[\hat\rho]}} . \tag{5.14}$$

Then one easily verifies that

$$\pi_{[\hat\rho\circ\rho]}(A)\, T_\rho \Big|_{\mathcal{H}_{[\hat\rho]}} \;=\; T_\rho\, \pi_{[\hat\rho]}\, (\rho(A)) \Big|_{\mathcal{H}_{[\hat\rho]}} . \tag{5.15}$$

We may now define field operators $\psi_\rho(B), B \in \mathcal{B}^{\mathcal{C}_a}$, by setting

$$\psi_\rho(B)\,\Phi \;=\; T_\rho\, \pi_{\hat\rho}(B)\,\Phi, \tag{5.16}$$

where Φ is an arbitrary vector in $\mathcal{H}_{[\hat\rho]}$.

These field operators are bounded versions of the more familiar unbounded field operators of Wightman field theory. They cannot, therefore, be strictly local. Their localization properties are described in the following proposition.

Proposition 5.5. Let $\rho \in \mathcal{L}_C$ and let $B \in \mathcal{A}(C)$, where C is some space-like cone. Then $\psi_\rho(B)$ commutes with all observables in $\mathcal{A}^c(C)$, (the subalgebra of $\mathcal{A}$ commuting with $\mathcal{A}(C)$).

Proof.

$$\begin{aligned} \pi_{[\hat\rho\circ\rho]}(A)\psi_\rho(B) &= \pi_{[\hat\rho\circ\rho]}(A)\, T_\rho\, \pi_{[\hat\rho]}(B) \\ &= T_\rho\, \pi_{[\hat\rho]}(\rho(A)\cdot B). \end{aligned} \tag{5.17}$$

Since $A \in \mathcal{A}^c(C)$, $\rho(A) = A$, as was shown in (4.3). Moreover, A commutes with B, since $B \in \mathcal{A}(C)$. Hence

$$\begin{aligned} \pi_{[\hat\rho\circ\rho]}(A)\, \psi_\rho(B) &= T_\rho\, \pi_{[\hat\rho]}(B)\, \pi_{[\hat\rho]}(A) \\ &= \psi_\rho(B)\, \pi_{[\hat\rho]}(A). \end{aligned}$$ ■

Remark. Since we shall always work with the total Hilbert space $\mathcal{H}_{tot.}$, defined in (5.8), and since by Proposition 5.1, (2) $\|\pi_{[\rho]}(A)\| = \|A\|$, for all $A \in \mathcal{A}$, where ρ is an arbitrary morphism in $\mathcal{L}_C$, and C is an arbitrary space-like cone, we shall, from now on, write A, instead of $\pi_{[\rho]}(A)$. Which representation of $\mathcal{A}$, or of $\mathcal{B}^{\mathcal{C}_a}$, A is to be evaluated in will be clear from context.

We now study further properties of the algebra of field operators. From equs. (5.16) and (5.17) it follows that

$$A\, \psi_\rho(B) \;=\; \psi_\rho(\rho(A)\cdot B), \tag{5.18}$$

and from (5.16) and (5.15)

$$\psi_{\rho_1}(B_1)\psi_{\rho_2}(B_2) \;=\; \psi_{\rho_2\circ\rho_1}(\rho_2^{\mathcal{C}_a}(B_1)\, B_2), \tag{5.19}$$

$\mathcal{C}_a \bigtimes (\mathcal{C}_1 \cup \mathcal{C}_2)$; ($\mathcal{C}_1$ and $\mathcal{C}_2$ are the cones where ρ_1 and ρ_2 are localized).

There is some redundance in our notion of field operators. Let $\rho_1 \in \mathcal{L}_{\mathcal{C}_1}$ and $\rho_2 \in \mathcal{L}_{\mathcal{C}_2}$ be equivalent morphisms of $\mathcal{B}^{\mathcal{C}_a}$, where the auxiliary cone is space-like to $\mathcal{C}_1 \cup \mathcal{C}_2$. Then there is a unitary intertwiner $\Gamma_{12} \equiv \Gamma(\mathcal{C}_1, \mathcal{C}_2)$ such that

$$\Gamma_{12}\, \rho_2(A) \;=\; \rho_1(A)\, \Gamma_{12}, \tag{5.20}$$

and, by property (C2), Γ_{12} can be chosen to belong to $\mathcal{B}(\mathcal{C}_1 \cup \mathcal{C}_2) \cap \overline{\mathcal{A}(\mathcal{S})}^w$, where $\mathcal{S}$ is a simple domain containing $\mathcal{C}_1$ and $\mathcal{C}_2$. We choose $\mathcal{C}_a$ such that $\mathcal{S} \subset \mathcal{C}_a'$. It is easy to check that for $\Phi \in \mathcal{H}_{[\hat{\rho}]}$, $\hat{\rho} \in \mathcal{L}_{\pi_0}$, and $B \in \mathcal{B}^{\mathcal{C}_a}$, the vectors

$$\psi_{\rho_1}(\Gamma_{12} B)\Phi \text{ and } \psi_{\rho_2}(B)\Phi \tag{5.21}$$

define the <u>same state</u> on $\mathcal{B}^{\mathcal{C}_a}$. For, for $A \in \mathcal{B}^{\mathcal{C}_a}$

$$\begin{aligned}
&< \psi_{\rho_1}(\Gamma_{12}B)\Phi, A\, \psi_{\rho_1}(\Gamma_{12}B)\Phi > \\
&\quad = < \psi_{\rho_1}(\Gamma_{12}B)\Phi,\ \psi_{\rho_1}(\rho_1^{\mathcal{C}_a}(A)\Gamma_{12}B)\Phi >, \text{ by (5.18)} \\
&\quad = < \psi_{\rho_1}(\Gamma_{12}B)\Phi,\ \psi_{\rho_1}(\Gamma_{12}\rho_2^{\mathcal{C}_a}(A)B)\Phi >, \text{ by (5.20)} \\
&\quad = < \Gamma_{12}B\Phi,\ \Gamma_{12}\rho_2^{\mathcal{C}_a}(A)B\Phi >, \text{ by (5.14), (5.16)} \\
&\quad = < B\Phi,\ \rho_2^{\mathcal{C}_a}(A)B\Phi >, \text{ because } \Gamma_{12} \text{ is unitary} \\
&\quad = < \psi_{\rho_2}(B)\Phi,\ \psi_{\rho_2}(\rho_2^{\mathcal{C}_a}(A)B)\Phi > \\
&\quad = < \psi_{\rho_2}(B)\Phi,\ A\,\psi_{\rho_2}(B)\Phi >, \text{ by (5.18)}
\end{aligned}$$

Of course, it does <u>not</u> follow from (5.21) that $\psi_{\rho_1}(\Gamma_{12}B)\Phi$ and $\psi_{\rho_2}(B)\Phi$ are the same vectors in $\mathcal{H}_{[\hat{\rho}\circ\rho_1]}$, much less that $\psi_{\rho_1}(\Gamma_{12}B)$ and $\psi_{\rho_2}(B)$ are the same operators. This observation will lead to a notion of field bundle which will display interesting topological properties; (see Sect. 6).

It is easy to extend the arguments in the proof of Proposition 5.5 to show that if $\Gamma_{12}B \in \overline{\mathcal{A}(\mathcal{C}_1)}^w$ then $\psi_{\rho_1}(\Gamma_{12}B)$ and $\psi_{\rho_2}(B)$ commute with all operators in $\mathcal{A}^c(\mathcal{C}_1)$.

<u>**Definition 5.6.**</u> A field $\psi_\rho(B)$ is said to be localized in a space-like cone $\mathcal{C} \subseteq \mathcal{C}_a'$ iff there is a unitary operator $\Gamma \in \mathcal{B}^{\mathcal{C}_a}$ intertwining ρ with $\hat{\rho}$ such that $\hat{\rho}$ is localized in $\mathcal{C}$ and $\Gamma B \in \overline{\mathcal{A}(\mathcal{C})}^w$. The familiy of fields localized in $\mathcal{C}$ is denoted by $\mathcal{F}^{\mathcal{C}_a}(\mathcal{C})$. ■

By the previous remark, any field in $\mathcal{F}^{\mathcal{C}_a}(\mathcal{C})$ commutes with all operators in $\mathcal{A}^c(\mathcal{C})$.

This definition is independent of the auxiliary cone $\mathcal{C}_a$, in the following sense.

<u>**Lemma 5.7.**</u> <u>Let $\psi_\rho(B) \in \mathcal{F}^{\mathcal{C}_a}(\hat{\mathcal{C}})$ be a field localized in $\hat{\mathcal{C}}$ and $\rho \in \mathcal{L}_{\mathcal{C}}$. If $\mathcal{S}$ is the minimal simple domain containing $\mathcal{C}$ and $\hat{\mathcal{C}}$ such that $\mathcal{S} \bigtimes \mathcal{C}_a$ and if $\hat{\mathcal{C}}_a$ is any other auxiliary cone satisfying $\hat{\mathcal{C}}_a \bigtimes \mathcal{S}$ then $\psi_\rho(B) \in \mathcal{F}^{\hat{\mathcal{C}}_a}(\hat{\mathcal{C}})$, or</u>

$$\mathcal{F}^{\mathcal{C}_a}(\hat{\mathcal{C}}) \;=\; \mathcal{F}^{\hat{\mathcal{C}}_a}(\hat{\mathcal{C}}).$$

<u>**Proof.**</u> Since $\psi_\rho(B)$ is localized in $\hat{\mathcal{C}} \subset \mathcal{C}_a'$, there exists a unitary intertwiner $\Gamma \in \mathcal{B}^{\mathcal{C}_a}$ such that

$$\Gamma\, \rho(A) \;=\; \hat{\rho}(A)\Gamma. \tag{5.22}$$

As the algebras $\overline{\mathcal{A}((\mathcal{C}_a + x)')}^w$ are norm dense in $\mathcal{B}^{\mathcal{C}_a}$ it is no loss of generality to assume that $\Gamma \in \overline{\mathcal{A}((\mathcal{C}_a + x)')}^w$ for some $x \in \mathrm{M}^3$ and, since $\mathcal{B}^{\mathcal{C}_a + x} = \mathcal{B}^{\mathcal{C}_a}$, we may set $x = 0$.

Property (C2) implies the existence of an intertwiner $\tilde{\Gamma}$ between ρ and $\hat{\rho}$ contained in $\mathcal{B}(\mathcal{C} \cup \hat{\mathcal{C}}) \cap \overline{\mathcal{A}(\mathcal{S})}^w$, so that

$$\tilde{\Gamma}\, \Gamma^* \hat{\rho}(A) \;=\; \hat{\rho}(A)\, \tilde{\Gamma}\, \Gamma^* \tag{5.23}$$

holds, by (5.22). Equations (4.3) and (4.9) imply that $\tilde{\Gamma}\,\Gamma^* \in \mathcal{B}(\hat{\mathcal{C}}) \cap \overline{\mathcal{A}(\mathcal{C}_a')}^w$ and hence $\tilde{\Gamma}\,\Gamma^* \in \overline{\mathcal{A}(\hat{\mathcal{C}})}^w$, by (C3). We conclude that

$$\tilde{\Gamma}\, B \;=\; (\tilde{\Gamma}\,\Gamma^*)(\Gamma\, B) \;\in\; \overline{\mathcal{A}(\hat{\mathcal{C}})}^w \tag{5.24}$$

and

$$B \;=\; \tilde{\Gamma}^*\, (\tilde{\Gamma}\, B) \;\in\; \overline{\mathcal{A}(\mathcal{S})}^w \;\subseteq\; \mathcal{B}^{\hat{\mathcal{C}}_a} \tag{5.25}$$

so that $\psi_\rho(B) \in \mathcal{F}^{\hat{\mathcal{C}}_a}(\hat{\mathcal{C}})$.

It remains to check that the action of the field $\psi_\rho(B)$ on $\mathcal{H}_{[\hat{\rho}]}$ is independent of the choice of the auxiliary cone $\mathcal{C}_a$, for all $\hat{\rho} \in \mathcal{L}_C$. That is,

$$\psi_{\hat{\rho}\circ\rho}(\hat{\rho}^{\mathcal{C}_a}(B)\hat{B}) = \psi_{\hat{\rho}\circ\rho}(\hat{\rho}^{\hat{\mathcal{C}}_a}(B)\hat{B}) \tag{5.26}$$

should hold for any $\mathcal{C}_a, \hat{\mathcal{C}}_a \times \mathcal{S}$ (see (5.19)). This is equivalent to

$$\hat{\rho}^{\mathcal{C}_a}(B) = \hat{\rho}^{\hat{\mathcal{C}}_a}(B), \ \forall \ \hat{\rho} \in \mathcal{L}_C, \ \mathcal{C}_a, \hat{\mathcal{C}}_a \times \mathcal{S}. \tag{5.27}$$

But $B \in \overline{\mathcal{A}(S)}^w \subseteq \mathcal{B}^{\mathcal{C}_a} \cap \mathcal{B}^{\hat{\mathcal{C}}_a}$ and hence (5.27) follows at once.

■

Remark: If ρ is irreducible, then equ. (5.23) implies that $\Gamma\,\tilde{\Gamma}^* \in \rho(\mathcal{A})'$ so that $\tilde{\Gamma} = \lambda\Gamma, |\lambda| = 1$ holds in $\mathcal{B}^{\mathcal{C}_a}$. The rest of the proof goes through without change, and we conclude that property (C3) may be omitted if we restrict ourselves to direct sums of irreducible representations.

6. The Field Bundle, and the Statistics of Fields.

In our analysis, the topology of the space, $\sum_3$, of space-like cones in three-dimensional Minkowski space plays a fairly important role; so we start by describing it. This will make us understand the basic differences between local, relativistic quantum theory in space-times of dimension three and those in four- or higher dimensional space-times.

Although the space $\sum_3$ of all space-like cones in M^3 is infinite-dimensional, its non-trivial topology resides in a simple finite-dimensional subspace: Let C be an open convex cone in the time $t = 0$ plane of M^3, whose boundary, ∂C, consists of two rays emanating from a point $x \in \mathsf{R}^2$, the apex of C. Let ε be the angle between the two rays in ∂C, and let r_C be the ray emanating from x and bisecting C into two equal pieces. Let α be the angle between r_C and the x^1-axis. Clearly, C is completely described by its apex x, its opening angle ε, and the angle α. Let $\mathcal{C} = (C')'$ be the causal completion of C. Then $\mathcal{C}$ is a space-like cone in M^3 with apex at x. It is completely determined by C and hence by the triple $(x,\varepsilon,\alpha) \in \mathsf{R}^2 \times (0,\pi) \times S^1$. Henceforth this will be the notion of space-like cone that we shall use.

Now the fundamental group of the space $\sum_3$ of space-like cones in M^3 is the same as the fundamental group of $\mathsf{R}^2 \times (0,\pi) \times S^1$. But

$$\pi_1(\mathsf{R}^2 \times (0,\pi) \times S^1) = \pi_1(S^1) = \mathsf{Z}. \tag{6.1}$$

Let us call α the asymptotic direction of $\mathcal{C}$, (or of the base, C, of $\mathcal{C}$). Then (6.1) says that, from the point of view of the fundamental group, $\sum_3$ is adequately described by the space of asymptotic directions which is the circle S^1. In higher dimensions, it is still true that the fundamental group of $\sum_{s+1}$ (the space of space-like cones in M^{s+1}) is described by the space of asymptotic directions, $\simeq S^{s-1}$, but $\pi_1(S^{s-1})$ is trivial for $s \geq 3$.

It is the non-trivial topology of $\sum_3$ which makes the problem of field statistics in three space-time dimensions more difficult than the statistics problem in higher dimensions and allows for the occurrence of braid statistics. Loosely speaking, braid statistics occurs when paths of field operators of the type constructed in Sect. 5 corresponding to loops ($\equiv$ closed paths), γ, in $\sum_3$ do not close which can happen when γ is not contractible. Non-contractible loops, γ, exist in $\sum_3$, because $\pi_1(\sum_3) = \pi_1(S^1) = \mathbb{Z}$, but there are no such loops in $\sum_{s+1}$, for $s \geq 3$, and this explains why statistics in local, relativistic quantum theories is conventional in space-times of dimension at least four.

Our purpose is now to make these remarks precise. We pick an auxiliary cone, $\mathcal{C}_a$, with asymptotic direction $\alpha_a \in (0, 2\pi)$ and an opening angle ε. We also choose a <u>reference cone</u>, $\mathcal{C}_0$, with asymptotic direction α_0 and an opening angle so small that $\mathcal{C}_0$ and $\mathcal{C}_a$ are space-like separated, i.e., $\mathcal{C}_0 \times \mathcal{C}_a$. Let $\pi \in \mathcal{L}_{\pi_0}$ be a representation of $\mathcal{A}$ localizable in cones relative to the vacuum representation π_0, and let $\rho \in \mathcal{L}_{\mathcal{C}_0}$ be a representation of $\mathcal{A}$ on $\mathcal{H}_0$ equivalent to π and localized in $\mathcal{C}_0$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be space-like cones such that $\mathcal{C}_1 \times \mathcal{C}_2$, and $(\mathcal{C}_1 \cup \mathcal{C}_2) \times \mathcal{C}_a$. Let $\rho_1 \in \mathcal{L}_{\mathcal{C}_1}$ and $\rho_2 \in \mathcal{L}_{\mathcal{C}_2}$ be representations unitary equivalent to ρ. By property (C2) of Sect. 4, see (4.12) and (4.13), there are unitary operators Γ_1 and Γ_2 such that

$$\Gamma_i^* \rho_i(A)\Gamma_i = \rho(A), \text{ for all } A \in \mathcal{A}, \tag{6.2}$$

and

$$\Gamma_i \in \mathcal{B}(\mathcal{C}_0 \cup \mathcal{C}_i) \cap \overline{\mathcal{A}(\mathcal{S}_i)}^w, \tag{6.3}$$

where $\mathcal{S}_i$ is a simple domain, (see (C1), Sect. 4), containing $\mathcal{C}_0 \cup \mathcal{C}_i$ with $\mathcal{S}_i \times \mathcal{C}_a$, for $i = 1, 2$. [The algebras $\mathcal{B}(\mathcal{C})$ have been defined in (4.9)].

Let $B_i \in \overline{\mathcal{A}(\mathcal{C}_i)}^w, i = 1, 2$. We set

$$\psi(\mathcal{C}_i, B_i) \equiv \psi_\rho(\Gamma_i^* B_i) = T_\rho \Gamma_i^* B_i, \tag{6.4}$$

where T_ρ is the isometry on $\mathcal{H}_{tot.}$ mapping $\mathcal{H}_{[\hat{\rho}]}$ to $\mathcal{H}_{[\hat{\rho}\circ\rho]}$, for any $\hat{\rho} \in \mathcal{L}_{\pi_0}$, constructed in (5.10)-(5.14). The second equation in (6.4) is definition (5.16) of $\psi_\rho(\cdot)$.

It is easy to see that $\psi(\mathcal{C}_i, B_i)$ belongs to the local field algebra $\mathcal{F}^{\mathcal{C}_a}(\mathcal{C}_i)$ introduced in Definition 5.6. Let Φ be a vector in $\mathcal{H}_{[\hat{\rho}]}$, for some $\hat{\rho} \in \mathcal{L}_{\pi_0}$. Then, by (5.10) or (5.15),

$$\begin{aligned}\psi(\mathcal{C}_1, B_1)\psi(\mathcal{C}_2, B_2)\Phi &= T_\rho \Gamma_1^* B_1 T_\rho \Gamma_2^* B_2 \Phi \\ &= T_\rho^2 \rho^{\mathcal{C}_a}(\Gamma_1^*)\rho^{\mathcal{C}_a}(B_1)\Gamma_2^* B_2 \Phi \\ &= T_{\rho\circ\rho}\rho^{\mathcal{C}_a}(\Gamma_1^*)\Gamma_2^* \, \rho_2^{\mathcal{C}_a}(B_1) B_2 \Phi,\end{aligned} \tag{6.5}$$

and we have used that $T_\rho^2 = T_{\rho\circ\rho}$. Since $B_i \in \overline{\mathcal{A}(\mathcal{C}_i)}^w$ and $\mathcal{C}_1 \times \mathcal{C}_2$,

$$\rho_2^{\mathcal{C}_a}(B_1) B_2 = B_1 B_2 = B_2 B_1 = \rho_1^{\mathcal{C}_a}(B_2) B_1. \tag{6.6}$$

Hence

$$\psi(\mathcal{C}_1, B_1)\,\psi(\mathcal{C}_2, B_2)\Phi = T_{\rho\circ\rho}\rho^{\mathcal{C}_a}(\Gamma_1^*)\Gamma_2^* \, B_2 B_1 \Phi. \tag{6.7}$$

Similarly, using again (6.6),

$$\psi(\mathcal{C}_2, B_2)\,\psi(\mathcal{C}_1, B_1)\Phi = T_{\rho\circ\rho}\rho^{\mathcal{C}_a}(\Gamma_2^*)\Gamma_1^* \, B_2 B_1 \Phi. \tag{6.8}$$

We define a <u>statistics operator</u>

$$\varepsilon_\rho^{\mathcal{C}_a}(\rho_1, \rho_2) = \rho^{\mathcal{C}_a}(\Gamma_1^*)\Gamma_2^*\Gamma_1 \, \rho^{\mathcal{C}_a}(\Gamma_2) \in \mathcal{B}^{\mathcal{C}_a}. \tag{6.9}$$

Then

$$\begin{aligned}T_{\rho\circ\rho}\varepsilon_\rho^{\mathcal{C}_a}(\rho_1, \rho_2)\, \rho^{\mathcal{C}_a}(\Gamma_2^*)\Gamma_1^* B_2 B_1 \Phi \\ = T_{\rho\circ\rho}\, \rho^{\mathcal{C}_a}(\Gamma_1^*)\Gamma_2^* B_2 B_1 \Phi.\end{aligned} \tag{6.10}$$

On the spaces $\mathcal{H}_{[\hat{\rho}\circ\rho\circ\rho]}$, $\hat{\rho} \in \mathcal{L}_{\pi_0}$, we may define the operator ("statistics matrix")

$$R_\rho^{\mathcal{C}_a}(\rho_1, \rho_2) = T_{\rho\circ\rho}\, \varepsilon_\rho^{\mathcal{C}_a}(\rho_1, \rho_2)\, T_{\rho\circ\rho}^{-1}. \tag{6.11}$$

By comparing (6.7) with (6.8) and using (6.10) and (6.11), we conclude that

$$\begin{aligned}&\psi(\mathcal{C}_1, B_1)\,\psi(\mathcal{C}_2, B_2)\Phi \\ &\quad = R_\rho^{\mathcal{C}_a}(\rho_1, \rho_2)\,\psi(\mathcal{C}_2, B_2)\,\psi(\mathcal{C}_1, B_1)\Phi.\end{aligned} \tag{6.12}$$

Thus the operators $\varepsilon_\rho^{C_a}(\rho_1,\rho_2)$ and $R_\rho^{C_a}(\rho_1,\rho_2)$ describe the commutation relations of space-like separated field operators. They have some fundamental properties described in the following theorem.

Theorem 6.1.

(1) Suppose that $\hat{C}_a$ is another auxiliary cone with the property that $\hat{C}_a \times S_i$, for $i = 1,2$. Then

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) \;=\; \varepsilon_\rho^{\hat{C}_a}(\rho_1,\rho_2).$$

(2) Suppose that $\hat{C}_i, i = 1,2$, are space-like cones with the property that the domains $\hat{C}_i \cup C_i$ are contained in simple domains $\hat{S}_i$, that

$$\hat{S}_i \times C_a, \text{ for } i = 1,2, \text{ and } \hat{S}_1 \times \hat{S}_2. \tag{6.13}$$

Let $\hat{\rho}_i \in \mathcal{L}_{\hat{C}_i}$ be unitary equivalent to ρ, for $i = 1,2$. Then

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) \;=\; \varepsilon_\rho^{C_a}(\hat{\rho}_1,\hat{\rho}_2). \tag{6.14}$$

(3) The operator $\varepsilon_\rho^{C_a}(\rho_1,\rho_2)$ commutes with $\rho^{C_a}(\rho^{C_a}(\mathcal{B}^{C_a}))$.

Proof. In order to prove (1), we note that since $C_a \times S_i$ and $\hat{C}_a \times S_i$, for $i = 1,2$, it follows that $\Gamma_1^* \in \overline{\mathcal{A}(S_1)}^w \subset \mathcal{B}^{C_a} \cap \mathcal{B}^{\hat{C}_a}, \Gamma_2 \in \overline{\mathcal{A}(S_2)}^w \subset \mathcal{B}^{C_a} \cap \mathcal{B}^{\hat{C}_a}$. Hence

$$\rho^{C_a}(\Gamma_1^*) \;=\; \rho^{\hat{C}_a}(\Gamma_1^*), \text{ and } \rho^{C_a}(\Gamma_2) \;=\; \rho^{\hat{C}_a}(\Gamma_2). \tag{6.15}$$

Thus (1) follows from (6.9) and (6.15).

The proofs of (2) and (3) are almost identical to the proofs of parts a) and c) of Lemma 2.6 in [7]. Since these are basic facts, where properties (C1)-(C3) must be used, we repeat the arguments.

Let $\hat{\Gamma}_i$ be unitary operators intertwining ρ and $\hat{\rho}_i \in \mathcal{L}_{\hat{C}_i}$, $i = 1,2$. Then

$$\hat{\Gamma}_i \;=\; \Delta_i\, \Gamma_i, \tag{6.16}$$

where Δ_i intertwines ρ_i and $\hat{\rho}_i$, for $i = 1,2$. Let $\tilde{S}_i$ be simple domains containing C_0 and $\hat{C}_i$, with $\tilde{S}_i \times C_a$, for $i = 1,2$. It follows from property (C2), Sect. 4, that

$$\Gamma_i \in \overline{\mathcal{A}(S_i)}^w, \text{ and } \hat{\Gamma}_i \in \overline{\mathcal{A}(\tilde{S}_i)}^w, \; i = 1,2.$$

Hence

$$\Delta_i = \hat{\Gamma}_i\Gamma_i^* \in \overline{\mathcal{A}(S_i \cup \tilde{S}_i)}^w, \; i = 1,2. \tag{6.17}$$

Since $\Delta_i \in \mathcal{B}(C_i \cup \hat{C}_i)$, by (4.13), and since $(S_i \cup \tilde{S}_i) \times C_a$ and $\hat{S}_i \times C_a$, it follows from (6.17) and property (C3), Sect. 4, that

$$\Delta_i \in \overline{\mathcal{A}(\hat{S}_i)}^w. \tag{6.18}$$

where $\hat{S}_i$ is a simple domain containing $C_i \cup \hat{C}_i$.

Remark. If ρ is irreducible then if Δ_i and $\tilde{\Delta}_i$ are two unitary operators intertwining ρ_i and $\hat{\rho}_i$ contained in an algebra $\overline{\mathcal{A}(S)}^w$, where S is a simple domain containing C_i and $\hat{C}_i$ and such that $S \times C_a$ then

$$\Delta_i \;=\; e^{i\theta}\, \tilde{\Delta}_i\,,$$

for some phase $e^{i\theta}$, see (4.16), and, since $\overline{\mathcal{A}(S)}^w \in \mathcal{B}^{C_a}$, it then follows from (C2) that

$$\Delta_i, \tilde{\Delta}_i \;\in\; \overline{\mathcal{A}(\hat{S}_i)}^w\,,$$

for any simple domain $\hat{S}_i$ containing $C_i \cup \hat{C}_i$, $i = 1,2$. Thus if ρ is irreducible we do not need (C3) to prove (6.18). Since, physically, only irreducible representations[5] are really important in this analysis, we learn that property (C3) could be omitted, provided one restricts all representations to be completely reducible!

We now conclude from (6.18) and locality that

$$\Delta_1\Delta_2 \;=\; \Delta_2\Delta_1, \; \rho_2^{C_a}(\Delta_1) \;=\; \Delta_2, \text{ and } \rho_1^{C_a}(\Delta_2) \;=\; \Delta_2.$$

[5] *They are expected to describe states of fixed "charge", in particular one-particle states.*

$$\begin{aligned}
\varepsilon_\rho^{C_a}(\hat\rho_1,\hat\rho_2) &= \rho^{C_a}(\Gamma_1^*)\rho^{C_a}(\Delta_1^*)\Gamma_2^*\Delta_2^*\Delta_1\Gamma_1\rho^{C_a}(\Delta_2)\rho^{C_a}(\Gamma_2)\\
&= \rho^{C_a}(\Gamma_1^*)\Gamma_2^*\rho_2^{C_a}(\Delta_1^*)\Delta_2^*\Delta_1\rho_1^{C_a}(\Delta_2)\Gamma_1\rho^{C_a}(\Gamma_2)\\
&= \rho^{C_a}(\Gamma_1^*)\Gamma_2^*\Delta_1^*\Delta_2^*\Delta_1\Delta_2\Gamma_1\rho^{C_a}(\Gamma_2)\\
&= \rho^{C_a}(\Gamma_1^*)\Gamma_2^*\Gamma_1\rho^{C_a}(\Gamma_2)\\
&= \varepsilon_\rho^{C_a}(\rho_1,\rho_2)
\end{aligned}$$

which proves (2).

Part (3) follows directly from the observation that $\psi(C_1,\mathbb{1})\psi(C_2,\mathbb{1})\Phi$ and $\psi(C_2,\mathbb{1})$-$\psi(C_1,\mathbb{1})\Phi$ define the same state on $\mathcal{A}$, by Proposition 5.4. Hence $\varepsilon_\rho^{C_a}(\rho_1,\rho_2)$ must commute with $\rho_1^{C_a}(\rho_2^{C_a}(\mathcal{B}^{C_a}))$ and hence with $\rho^{C_a}(\rho^{C_a}(\mathcal{B}^{C_a}))$.

■

Next, we should ask how $\varepsilon_\rho^{C_a}(\rho_1,\rho_2)$ depends on the choice of the reference morphism ρ. Suppose that ρ is replaced by an equivalent morphism $\tilde\rho$ localized in a cone $\tilde C$, with $\tilde C \times C_a$. Let ρ_1 and ρ_2 be as above, and let $\tilde\Gamma$ be a unitary operator intertwining ρ and $\tilde\rho$, i.e.

$$\tilde\rho(A) \;=\; \tilde\Gamma^*\,\rho(A)\tilde\Gamma,\;\; A\in\mathcal{A}, \tag{6.19}$$

with $\tilde\Gamma \in \mathcal{B}^{C_a}$. Then

$$\varepsilon_{\tilde\rho}^{C_a}(\rho_1,\rho_2) \;=\; (\rho^{C_a}(\tilde\Gamma)\tilde\Gamma)^*\,\varepsilon_\rho^{C_a}(\rho_1,\rho_2)\rho^{C_a}(\tilde\Gamma)\tilde\Gamma, \tag{6.20}$$

as follows easily from (6.19) and (6.9), using that $\tilde\rho^{C_a}(A) = \tilde\Gamma^*\Gamma_i^*\rho_i^{C_a}(A)\Gamma_i\tilde\Gamma$, for $A\in\mathcal{A}$.

Let us fix the auxiliary cone C_a. Without loss of generality, we may suppose that the asymptotic direction, α_a, of C_a is

$$\alpha_a \;=\; \pi. \tag{6.21}$$

Given some morphism $\rho\in\mathcal{L}_C$ we define the asymptotic direction, as (ρ), of ρ to be the asymptotic direction of the space-like cone C in which ρ is localized. Let ε be the

opening angle of C_a. In accordance with (6.21) we require that

$$-\pi + \frac{\varepsilon}{2} < \text{ as } (\rho) < \pi - \frac{\varepsilon}{2}. \tag{6.22}$$

Consider a representation $\pi\in\mathcal{L}_{\pi_0}$, and pick a morphism ρ localized in a space-like cone C_0, with $C_0 \times C_a$ such that π is equivalent to ρ. Let ρ_1 and ρ_2 be equivalent to ρ and localized in space-like cones C_1 and C_2, respectively, with $C_i \times C_a$, $i=1,2$, and $C_1 \times C_2$, as assumed in Theorem 6.1.

__Definition 6.2.__ We define

$$\varepsilon_\rho^{C_a,>} \;\equiv\; \varepsilon_\rho^{C_a}(\rho_1,\rho_2) \text{ if } as(\rho_1) > as(\rho_2), \tag{6.23}$$

and

$$\varepsilon_\rho^{C_a,<} \;\equiv\; \varepsilon_\rho^{C_a}(\rho_1,\rho_2) \text{ if } as(\rho_2) > as(\rho_1). \tag{6.24}$$

This definition is illustrated in the following Fig. 7, (b).

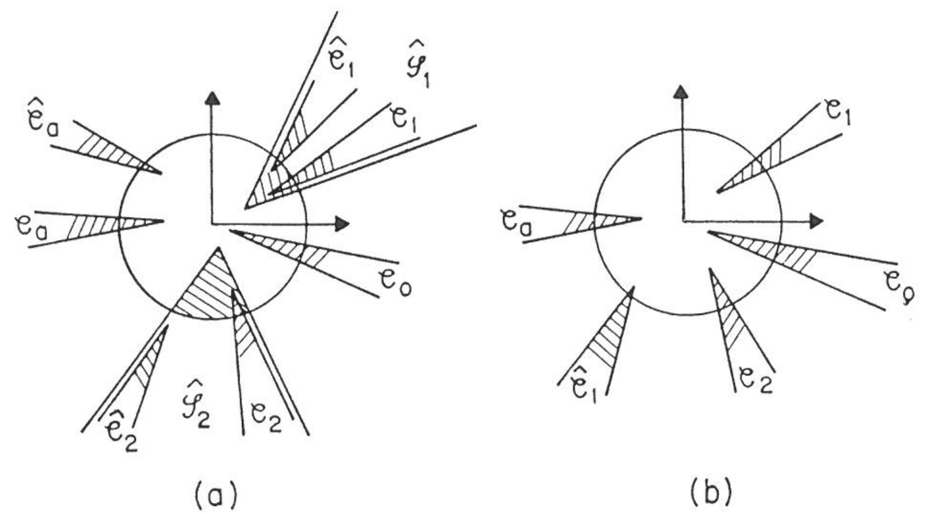

Fig. 7

Suppose that $as(\rho_1) > as(\rho_2)$. Let $\hat{\rho}_i$ be equivalent to ρ_i, and such that the hypotheses of part (2) of Theorem 6.1 are satisfied. Then Fig. 2, (a) shows that $as(\hat{\rho}_1) > as(\hat{\rho}_2)$, and Theorem 6.1, (2) tells us that

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) = \varepsilon_\rho^{C_a}(\hat{\rho}_1,\hat{\rho}_2).$$

Moreover, within the limits specified in part (2) of Theorem 6.1 and indicated in Fig. 2, (a)

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) = \varepsilon_\rho^{\hat{C}_a}(\rho_1,\rho_2).$$

Using now part (2) of Theorem 6.1 to move ρ_1 to $\hat{\rho}_1, \rho_2$ to $\hat{\rho}_2$, with $as(\hat{\rho}_1) > as(\hat{\rho}_2)$, and then again part (1), and so on, we see that, as long as $C_a \times C_0$, $\varepsilon_\rho^{C_a,>}$ is well defined and independent of C_a. Thus, we may write

$$\varepsilon_\rho^{C_a,>} =: \varepsilon_\rho^{>}, \tag{6.25}$$

and similarly $\varepsilon_\rho^{C_a,<} =: \varepsilon_\rho^{<}$.

For $C_0, C_1, \hat{C}_1, C_2, C_a$ as in Fig. 2, (b), a simple domain containing $C_1 \cup \hat{C}_1$ cannot be space-like separated from C_2 and from C_a, so that part (2) of Theorem 6.1 does not apply. In general,

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) = \varepsilon_\rho^{>} \neq \varepsilon_\rho^{<} = \varepsilon_\rho^{C_a}(\hat{\rho}_1,\rho_2).$$

We define

$$R_\rho^{\#} = T_{\rho_0\rho}\varepsilon_\rho^{\#}T_{\rho_0\rho}^{-1}, \quad \text{for } \# =>, \text{ or } <. \tag{6.26}$$

We summarize our discussion in a theorem.

Theorem 6.3.

Let $\rho \in [\rho] \in \mathcal{L}_{\pi_0}$. Then ρ determines two unitary operators, $\varepsilon_\rho^{>} \in \mathcal{B}^{C_a}$ and $\varepsilon_\rho^{<} \in \mathcal{B}^{C_a}$, where C_a is any auxiliary cone in the space-like complement of C_0, the localization cone of ρ, such that

(1) $\varepsilon_\rho^{>}$ and $\varepsilon_\rho^{<}$ are independent of C_a;

(2) $\varepsilon_\rho^{>}$ and $\varepsilon_\rho^{<}$ commute with $\rho^{C_a}(\rho(\mathcal{A}))$;

(3) $\varepsilon_\rho^{>} \cdot \varepsilon_\rho^{<} = 1\!\!1$. (6.27)

(4) If ρ is equivalent to $\tilde{\rho}$ then

$$\varepsilon_{\tilde{\rho}}^{>} = U^*\varepsilon_\rho^{>}U, \quad \varepsilon_{\tilde{\rho}}^{<} = U^*\varepsilon_\rho^{<}U,$$

for some unitary operator $U \in \mathcal{B}^{C_a}$ only depending on ρ and $\tilde{\rho}$; and

(5) if $\psi(C_i,B_i) \equiv \psi_\rho(C_i,B_i)$, $i = 1,2$, are the field operators defined in (6.4) then

$$\psi(C_1,B_1)\psi(C_2,B_2) = R_\rho^{\gtrless}\psi(C_2,B_2)\psi(C_1,B_1), \tag{6.28}$$

for $as(C_1) \gtrless as(C_2)$, where $as(C)$ is the asymptotic direction of the space-like cone C relative to the auxiliary cone C_a.

Proof. (1) follows from parts (1) and (2) of Theorem 6.1 and the remarks above; see (6.25). In view of Definition 6.2, (2) is identical to part (3) of Theorem 6.1. Part (3) is a simple calculation, using (6.9) and (6.23),(6.24): if $as(\rho_1) > as(\rho_2)$ then

$$\begin{aligned}\varepsilon_\rho^{>} \cdot \varepsilon_\rho^{<} &= \varepsilon_\rho^{C_a}(\rho_1,\rho_2)\varepsilon_\rho^{C_a}(\rho_2,\rho_1)\\ &= \rho^{C_a}(\Gamma_1^*)\Gamma_2^*\Gamma_1\rho^{C_a}(\Gamma_2)\rho^{C_a}(\Gamma_2^*)\Gamma_1^*\Gamma_2\rho^{C_a}(\Gamma_1)\\ &= 1\!\!1.\end{aligned}$$

Part (4) is (6.20), (with $U = \rho^{C_a}(\bar{\Gamma})\tilde{\Gamma} \in \mathcal{B}^{C_a}$), and (5) follows from (6.12) and the definition of $R_\rho^{\gtrless}$; see (6.26).

■

Next, we wish to relate $\varepsilon_\rho^{>}$ to $\varepsilon_\rho^{<}$. Consider the geometrical situation depicted in Fig. 8:

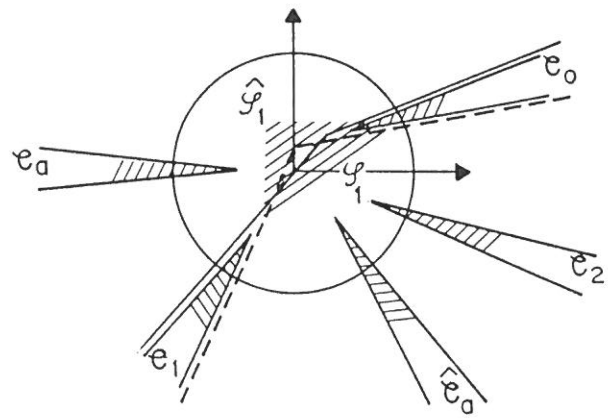

Fig. 8

Then we have

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) = \varepsilon_\rho^<, \text{ and}$$
$$\varepsilon_\rho^{\hat{C}_a}(\rho_1,\rho_2) = \varepsilon_\rho^> . \tag{6.29}$$

By (6.9),

$$\varepsilon_\rho^{C_a}(\rho_1,\rho_2) = \rho^{C_a}(\Gamma_1^*)\Gamma_2^*\Gamma_1\,\rho^{C_a}(\Gamma_2)$$
$$\varepsilon_\rho^{\hat{C}_a}(\rho_1,\rho_2) = \rho^{\hat{C}_a}(\hat{\Gamma}_1^*)\Gamma_2^*\hat{\Gamma}_1\,\rho^{\hat{C}_a}(\Gamma_2), \tag{6.30}$$

where $\Gamma_2 \in \overline{\mathcal{A}(\mathcal{S}_2)}^w \subset \mathcal{B}^{C_a} \cap \mathcal{B}^{\hat{C}_a}, \mathcal{S}_2$ is a simple domain containing $\overline{C_0 \cup C_2}^w$ space-like to C_a and $\hat{C}_a, \Gamma_1 \in \overline{\mathcal{A}(\mathcal{S}_1)}^w \subset \mathcal{B}^{C_a}$, but $\Gamma_1 \notin \mathcal{B}^{\hat{C}_a}$, and $\hat{\Gamma}_1 \in \overline{\mathcal{A}(\hat{\mathcal{S}}_1)}^w \subset \mathcal{B}^{\hat{C}_a}$, but $\hat{\Gamma}_1 \notin \mathcal{B}^{C_a}$. [By (C3), $\mathcal{S}_1$ is the minimal simple domain containing $C_0 \cup C_1$ and space-like to C_a, while $\hat{\mathcal{S}}_1$ is space-like to $\hat{C}_a$.] Of course, Γ_1 and $\hat{\Gamma}_1$ may, in general, be different unitary operators.

Since $\rho^{C_a}(\Gamma_2) = \rho^{\hat{C}_a}(\Gamma_2)$, it follows from (6.29), (6.30) and (6.2) that

$$\varepsilon_\rho^< = \rho^{C_a}(\Gamma_1^*)\Gamma_2^*\rho_1^{C_a}(\Gamma_2)\Gamma_1$$
$$= \rho^{C_a}(\Gamma_1^*)\rho^{\hat{C}_a}(\hat{\Gamma}_1)\varepsilon_\rho^>\hat{\Gamma}_1^*\Gamma_1 . \tag{6.31}$$

It is easy to see that, as operators defined on $\mathcal{H}_0$, $\Gamma_1^*\hat{\Gamma}_1$ commutes with $\rho(\mathcal{A})$ and $\rho^{C_a}(\Gamma_1^*)\rho^{\hat{C}_a}(\hat{\Gamma}_1)$ commutes with $\rho^{C_a}(\rho(\mathcal{A}))$ $(= \rho^{\hat{C}_a}(\rho(\mathcal{A})))$. If ρ is irreducible then $\rho(\mathcal{A})' = \{\lambda\mathbb{I} : \lambda \in \mathbb{C}\}$, and it follows that

$$\Gamma_1^*\hat{\Gamma}_1 = e^{2\pi i\, s_\rho}, \quad \text{on } \mathcal{H}_0, \tag{6.32}$$

where s_ρ is some real number. We shall see in Sect. 8 that s_ρ is the spin *mod* $\mathbb{Z}$ of the states in $\mathcal{H}_{[\rho]}$.

The key observation is now that, since there is no auxiliary cone space-like to $\mathcal{S}_1$ and to $\hat{\mathcal{S}}_1$, $\hat{\Gamma}_1^*\Gamma_1$ is not an element of any $\mathcal{B}^{\tilde{C}_a}$, for any auxiliary cone $\tilde{C}_a$, and hence $\rho^{C_a}(\Gamma_1^*)\rho^{\hat{C}_a}(\hat{\Gamma}_1)$ will, in general, not be given by $\rho^{\tilde{C}_a}(\Gamma_1^*\hat{\Gamma}_1)$; in particular, it need not be given by $e^{2\pi i s_\rho}$. Thus we conclude that, in general, $\varepsilon_\rho^<$ and $\varepsilon_\rho^>$ may be distinct unitary operators. This is the origin of braid statistics in three-dimensional local relativistic quantum theories, and our arguments have illustrated its simple topological origin. This will be elaborated upon in Sect. 8.

Next, we derive commutation relations between two fields

$$\psi_\rho(\Gamma^*B_1) = T_\rho\Gamma^*B_1 \text{ and } \psi_{\tilde{\rho}}(\tilde{\Gamma}^*B_2) = T_{\tilde{\rho}}\tilde{\Gamma}^*B_2, \tag{6.33}$$

where Γ and $\tilde{\Gamma}$ are unitary intertwiners such that

$$\rho_1(A) = \Gamma\rho(A)\Gamma^*, \quad \rho_2(A) = \tilde{\Gamma}\tilde{\rho}(A)\tilde{\Gamma}^*, \tag{6.34}$$

for $A \in \mathcal{A}$; ρ_i is localized in a space-like cone C_i, and $B_i \in \overline{\mathcal{A}(C_i)}^w$, for $i = 1,2$: moreover, $C_1 \times C_2$. [See (5.16) and (6.4).] It is no longer assumed that the morphisms ρ and $\tilde{\rho}$ are unitary equivalent! In this case we shall encounter some ambiguity, as one would expect from local quantum field theory.

It is convenient to choose the reference morphisms ρ and $\tilde{\rho}$ to be localized in space-like separated space-like cones C and $\tilde{C}$, respectively, sucht that $C \cup \tilde{C} \times C_a$, for some auxiliary cone C_a. By proposition 5.4,

$$\rho^{C_a} \circ \tilde{\rho}^{C_a} = \tilde{\rho}^{C_a} \circ \rho^{C_a}, \quad \text{on } \mathcal{B}^{C_a}. \tag{6.35}$$

Thanks to (6.35) one can construct the isometries T_ρ and $T_{\tilde\rho}$ on $\mathcal{H}_{tot.}$ to <u>commute</u>, i.e.,

$$T_\rho T_{\tilde\rho} \;=\; T_{\tilde\rho} T_\rho. \tag{6.36}$$

However, since T_ρ is not uniquely fixed by ρ, other choices for T_ρ and $T_{\tilde\rho}$ are conceivable for which $[T_\rho, T_{\tilde\rho}] \neq 0$. This ambiguity has been analyzed in local quantum field theory by Araki [35]. For concreteness we shall impose (6.36) and suppose that

$$as(\rho) \;>\; as(\tilde\rho), \tag{6.37}$$

relative to C_a.

We define a statistics operator

$$\varepsilon^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2) \;=\; \tilde\rho^{C_a}(\Gamma^*)\tilde\Gamma^*\Gamma\,\rho^{C_a}(\tilde\Gamma). \tag{6.38}$$

By (6.34), $\varepsilon^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2)$ is equal to

$$\varepsilon^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2) \;=\; \tilde\Gamma^*\rho_2^{C_a}(\Gamma^*)\rho_1^{C_a}(\tilde\Gamma)\Gamma. \tag{6.39}$$

The statistics matrix, R, is defined by

$$R^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2) \;=\; T_{\tilde\rho}T_\rho\,\varepsilon^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2)(T_{\tilde\rho}T_\rho)^{-1} \tag{6.40}$$

Repeating calculations (6.5) - (6.12), we find that

$$\psi_\rho(\Gamma^*B_1)\psi_{\tilde\rho}(\tilde\Gamma^*B_2)$$
$$= R^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2)\,\psi_{\tilde\rho}(\tilde\Gamma^*B_2)\,\psi_\rho(\Gamma^*B_1) \tag{6.41}$$

on $\mathcal{H}_{tot.}$. This is the analogue of (6.12).

Using properties (C1) - (C3), Sect. 4, and equs. (6.38), (6.39), it is straightforward to extend Theorems 6.1 and 6.3 to the present situation; (this is a useful exercise for the reader). Thus, for $as(\rho_1) \gtrless as(\rho_2)$, relative to some auxiliary cone C_a space-like separated from $C, \tilde C, C_1$ and C_2, we define

$$\varepsilon^{\gtrless}_{\rho,\tilde\rho} \;\equiv\; \varepsilon^{C_a}_{\rho,\tilde\rho}(\rho_1,\rho_2), \tag{6.42}$$

and

$$R^{\gtrless}_{\rho,\tilde\rho} \;=\; T_{\tilde\rho}T_\rho\,\varepsilon^{\gtrless}_{\rho,\tilde\rho}(T_{\tilde\rho}T_\rho)^{-1}. \tag{6.43}$$

Then one has the following theorem.

<u>**Theorem 6.4.**</u>

(1) $\varepsilon^{>}_{\rho,\tilde\rho}$ and $\varepsilon^{<}_{\rho,\tilde\rho}$ <u>are independent of</u> C_a.

(2) $\varepsilon^{\gtrless}_{\rho,\tilde\rho}$ <u>commute with</u> $\rho^{C_a}(\tilde\rho^{C_a}(\mathcal{B}^{C_a})) = \tilde\rho^{C_a}(\rho^{C_a}(\mathcal{B}^{C_a}))$.

(3) $\varepsilon^{>}_{\rho,\tilde\rho}\,\varepsilon^{<}_{\tilde\rho,\rho} \;=\; \mathbb{1}$.

(4) <u>If,</u> ρ <u>is equivalent to</u> ρ' <u>and</u> $\tilde\rho$ <u>is equivalent to</u> $\tilde\rho'$ <u>then</u>

$$\varepsilon^{\gtrless}_{\rho',\tilde\rho'} \;=\; U^*\,\varepsilon^{\gtrless}_{\rho,\tilde\rho}\,U,$$

<u>for some unitary operator</u> $U \in \mathcal{B}^{C_a}$ <u>only depending on</u> $\rho,\tilde\rho,\rho'$ <u>and</u> $\tilde\rho'$.

(5) $\psi_\rho(\Gamma^*B_1)\psi_{\tilde\rho}(\tilde\Gamma^*B_2) = R^{\gtrless}_{\rho,\tilde\rho}\psi_{\tilde\rho}(\tilde\Gamma^*B_2)\psi_\rho(\Gamma^*B_1)$, <u>for</u> $as(C_1) \gtrless as(C_2)$.

(6) <u>With the obvious meaning of</u> $\Gamma, \hat\Gamma, C_a$ <u>and</u> $\hat C_a$,

$$\varepsilon^{<}_{\rho,\tilde\rho} \;=\; \tilde\rho^{C_a}(\Gamma^*)\tilde\rho^{\hat C_a}(\hat\Gamma)\varepsilon^{>}_{\rho,\tilde\rho}\hat\Gamma^*\Gamma.$$

□

<u>**Remark.**</u> Since we have assumed that $C \times \tilde C$, with $as(\rho) \equiv as(C) > as(\tilde C) \equiv as(\tilde\rho)$, see (6.37), we may set $\rho_1 = \rho, C_1 = C$ and $\rho = \tilde\rho, C_2 = \tilde C$, so that $\Gamma = \tilde\Gamma = \mathbb{1}$. It then follows from (6.42) and (6.38) that

$$\varepsilon^{>}_{\rho,\tilde\rho} \;=\; \mathbb{1}, \tag{6.44}$$

hence, by part (3) of Theorem 6.4,

$$\varepsilon^{<}_{\rho,\tilde\rho} \;=\; \mathbb{1}.$$

However, $\varepsilon^{<}_{\rho,\tilde\rho} = (\varepsilon^{>}_{\tilde\rho,\rho})^{-1}$, will, in general be different from $\mathbb{1}$. If we replaced (6.37) by the condition that $as(\rho) < as(\tilde\rho)$, then (6.44) is replaced by

$$\varepsilon^{<}_{\rho,\tilde\rho} \;=\; \mathbb{1}, \quad (\text{but } \varepsilon^{>}_{\rho,\tilde\rho} \neq \mathbb{1}). \tag{6.45}$$

Replacing (6.36) by

$$T_\rho T_{\bar\rho} \;=\; T_{\bar\rho} T_\rho \,\varepsilon,$$

for some unitary operator $\varepsilon \in \mathcal{B}^{\mathcal{C}_a}$ commuting with $\rho^{\mathcal{C}_a}(\bar\rho(\mathcal{A}))$, we can achieve that $\varepsilon^{>}_{\rho,\bar\rho} \neq \mathbb{I}$ and $\varepsilon^{<}_{\rho,\bar\rho} \neq \mathbb{I}$.

Thus the statistics operators $\varepsilon^{\gtrless}_{\rho,\bar\rho}$ depend on various conventions. Objects which are invariantly associated with $[\rho]$ and $[\bar\rho]$, up to conjugation by unitary operators in $\mathcal{B}^{\mathcal{C}_a}$, are the monodromy operators,

$$\mu^{\gtrless}_{\rho,\bar\rho} \;=\; \varepsilon^{\gtrless}_{\rho,\bar\rho}\,\varepsilon^{\gtrless}_{\bar\rho,\rho}, \tag{6.46}$$

in contrast to the "half-monodromies", $\varepsilon^{\gtrless}_{\rho,\bar\rho}$.

7. Representations of the Braid Groups, Statistical Dimension and (Charge-) Conjugate Sectors.

In this section, we show that the statistics operators $\varepsilon^{>}_{\rho}$, $\varepsilon^{<}_{\rho}$ determine unitary representations of the braid groups, B_n, on n strands, $n = 2,3,4,\ldots$. These representations will turn out to describe the statistics of multi-particle wave functions, (describing the state of asymptotic charged particles); see Sect. 9.

To begin with, let us recall the definition of the braid groups: The braid group, B_n, on n strands can be defined by its generators $\tau_1, \ldots, \tau_{n-1}$, satisfying the relations

$$\tau_i\,\tau_{i+1}\,\tau_i \;=\; \tau_{i+1}\,\tau_i\,\tau_{i+1}, \tag{7.1}$$

for $i = 1, \ldots, n-2$, and

$$\tau_i\,\tau_j \;=\; \tau_j\,\tau_i, \quad \text{for} \quad |\,i-j\,| \geq 2\,. \tag{7.2}$$

The inverse of τ_i is denoted by τ_i^{-1}, for all i, and the identity element of B_n is denoted by 1. The center of B_n is generated by $(\tau_1\ \tau_2\ \ldots\ ,\tau_{n-1})^n$.

Given n vertical strands in $\mathbb{R}^3$,

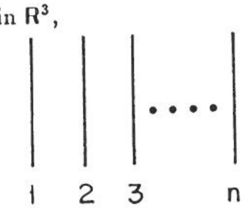

Fig. 9

element τ_i acts on these strands by braiding the i^{th} strand once:

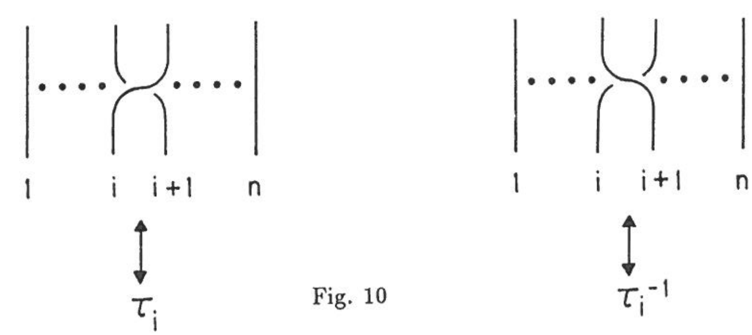

Fig. 10

General diagrams with many braided strands are called braids; for precise definitions see e.g. [36]. Multiplication in B_n can be described as the composition of braid diagrams and rearranging the strands by ambient isotopies of $\mathbb{R}^3$ with the property that distinct strands never intersect. These rearrangements of strands precisely correspond to rearranging words in $\{\tau_1^{\pm 1}, \dots, \tau_{n-1}^{\pm 1}\}$ by using the relations (7.1) and (7.2).

The groupoid on n coloured strands, B_n^c, is obtained from B_n by assigning different colours to all strands and requiring that two n-coloured braid diagrams be composable only if the colours of the strands match. Clearly, B_n acts on B_n^c in the obvious way.

Let $\mathring{\rho}_1, \dots, \mathring{\rho}_n$ be n (possibly inequivalent) morphisms of the algebra $\mathcal{B}^{C_a}$ localized in space-like cones $\mathring{C}_1, \dots \mathring{C}_n$, with the property that $\mathring{C}_i \times \mathring{C}_j$, for $i \neq j$, and $\mathring{C}_i \times C_a$, for some auxiliary space-like cone $C_a, i = 1, \dots, n$. Let Γ_i be unitary intertwiners such that

$$\rho_i(A) = \Gamma_i \mathring{\rho}_i(A) \Gamma_i^*, \quad A \in \mathcal{A}, \; i = 1, \dots, n, \tag{7.3}$$

and such that ρ_i is localized in C_i, with $C_i \times C_j$, for $i \neq j$, and $C_i \times C_a$. If $\mathring{\rho}_1, \dots, \mathring{\rho}_n$ are all equivalent to some $\mathring{\rho} \in [\mathring{\rho}] \in \mathcal{L}_{\pi_0}$ then we set $\mathring{\rho}_1 = \dots = \mathring{\rho}_n = \mathring{\rho}$, ($\mathring{C}_1 = \dots = \mathring{C}_n = \mathring{C}$, with $\mathring{C} \times C_a$). We set

$$\psi_i(C_i) \;=\; T_{\mathring{\rho}_i} \; \Gamma_i^* \; B_i, \tag{7.4}$$

with $B_i \in \overline{\mathcal{A}(C_i)}^w$. Since $C_i \times C_j$, $\rho^{C_a}(B_i)$ commutes with $\tilde{\rho}^{C_a}(B_j)$, for $i \neq j$ and arbitrary morphisms, ρ^{C_a} and $\tilde{\rho}^{C_a}$, obtained by composing some of the morphisms $\rho_1^{C_a}, \dots \rho_n^{C_a}$; (here we are using (4.17) and (4.3)). The statistics (commutation relations) of the fields $\psi_i(C_i)$ is therefore independent of the operators B_i, and hence we may choose, for simplicity, $B_i = \mathbb{1}$, for all i.

We now consider the product $\psi_1(C_1) \dots \psi_n(C_n)$ and study the effect of interchanging $\psi_i(C_i)$ with $\psi_{i+1}(C_{i_1})$. By (7.4), for $B_i = \mathbb{1}$, and (6.36), i.e., $T_{\mathring{\rho}_i}$ and $T_{\mathring{\rho}_{i+1}}$

commute, we obtain, as in (6.10), that

$$\begin{aligned} \psi_i(C_i)\psi_{i+1}(C_{i+1}) &= T_{\mathring{\rho}_{i+1}} \; T_{\mathring{\rho}_i} \; \varepsilon^{\#}_{\mathring{\rho}_i \mathring{\rho}_{i+1}} \; \mathring{\rho}_i^{C_a}(\Gamma_{i+1}^*)\Gamma_i^* \\ &= R^{\#}_{\mathring{\rho}_i \mathring{\rho}_{i+1}} \; \psi_{i+1}(C_{i+1})\,\psi_i(C_i), \end{aligned} \tag{7.5}$$

where $\# = \gtrless$, for as $(C_i) \gtrless$ as (C_{i+1}), and $R^{\gtrless}_{\mathring{\rho}_i \mathring{\rho}_{i+1}}$ is defined in (6.43). By part (2) of Theorem 6.4, $\varepsilon^{\gtrless}_{\mathring{\rho}_i \mathring{\rho}_{i+1}}$ commutes with $\mathring{\rho}_{i+1}^{C_a}(\mathring{\rho}_i^{C_a}(\mathcal{B}^{C_a}))$. Hence $\varepsilon^{\gtrless}_{\mathring{\rho}_i \mathring{\rho}_{i+1}}$ commutes with all operators

$$\mathring{\rho}_{i+1}^{C_a}(\mathring{\rho}_i^{C_a}(\rho^{C_a}(\Gamma_j^*))), \quad \text{for } j < i,$$

where ρ^{C_a} is any morphism obtained by composing some of the morphisms $\rho_1^{C_a}, \dots, \rho_{i-1}^{C_a}$. From this it follows immediately that

$$\begin{aligned} &\psi_1(C_1) \dots \psi_i(C_i)\,\psi_{i+1}(C_{i+1}) \dots \psi_n(C_n) \\ &\quad = R_i^{\gtrless}(\mathring{\rho}_1, \dots, \mathring{\rho}_n)\,\psi_1(C_1) \dots \psi(C_{i+1})\,\psi(C_i) \dots \psi_n(C_n), \end{aligned} \tag{7.6}$$

if as $(C_i) \gtrless$ as (C_{i+1}), where

$$R_i^{\gtrless}(\mathring{\rho}_1, \dots, \mathring{\rho}_n) \equiv T_{\mathring{\rho}_1} \dots T_{\mathring{\rho}_{i+1}} \; \varepsilon_{\mathring{\rho}_i \mathring{\rho}_{i+1}} \; T^{-1}_{\mathring{\rho}_{i+1}} \dots T^{-1}_{\mathring{\rho}_1}. \tag{7.7}$$

Let b be an element of B_n presented as a word in the generators $\{\tau_1^{\pm 1}, \dots, \tau_n^{\pm 1}\}$, i.e.,

$$b = \tau_{i_k}^{\varepsilon_k} \dots \tau_{i_1}^{\varepsilon_1}, \quad \text{for some } k \in \mathbb{N}, \tag{7.8}$$

where $\varepsilon_j = \pm 1$, $i_j \in \{1, \dots, n-1\}$, for all $j = 1, \dots, k$. For $1 \leq m < k$, we set $b_m = \tau_{i_m}^{\varepsilon_m} \dots \tau_{i_1}^{\varepsilon_1}$, and we define π_m to be the permutation of the endpoints of the n coloured strands corresponding to the element $b_m \in B_n$, with $\tau_0 \equiv id$. We define

$$R(\tau_i^{\pm 1}, \pi) \;\equiv\; R_i^{\gtrless}(\mathring{\rho}_{\pi(1)}, \dots, \mathring{\rho}_{\pi(n)}). \tag{7.9}$$

Let $\hat{\rho}$ be an arbitrary localized morphism, i.e., $[\hat{\rho}] \in \mathcal{L}_{\pi_0}$, and set

$$\mathcal{H} \;\equiv\; \mathcal{H}_{[\hat{\rho} \circ \mathring{\rho}_n \dots \circ \mathring{\rho}_1]}. \tag{7.10}$$

For b given by (7.8), we set

$$R(b;\mathring{\rho}_1,\ldots,\mathring{\rho}_n) \;=\; \prod_{m=k}^{1} R(\tau_{i_m}^{\epsilon_m},\pi_{m-1}) . \tag{7.11}$$

Proposition 7.1.

The assignment

$$B_n \ni b \longrightarrow R(b;\mathring{\rho}_1,\ldots,\mathring{\rho}_n) \tag{7.12}$$

defines a unitary representation of the groupoid, B_n^c, of n-coloured strands on the Hilbert-space $\mathcal{H}$.

Proof. Clearly the opertors $R_i^{\gtrless}(\mathring{\rho}_{\pi(1)},\ldots,\mathring{\rho}_{\pi(n)})$ and hence the operators $R(b;\mathring{\rho}_1,\ldots,\mathring{\rho}_n)$ are unitary operators on $\mathcal{H}$, for all $b\in B_n$. We must show that R respects relations (7.1) and (7.2): let σ_j denote transposition of j with $j+1$. Then (7.1) would imply that

$$\begin{aligned} &R(\tau_i,\sigma_{i+1}\circ\sigma_i\circ\pi)\; R(\tau_{i+1},\sigma_i\circ\pi)\; R(\tau_i,\pi),\\ &= R(\tau_{i+1},\sigma_i\circ\sigma_{i+1}\circ\pi)\; R(\tau_i,\sigma_{i+1}\circ\pi)\; R(\tau_{i+1},\pi), \end{aligned} \tag{7.13}$$

for $i=1,\ldots,n-2$, and every permutation π of $\{1,\ldots,n\}$. Equ. (7.13) follows immediately from (7.6), (7.7) and the associativity of multiplication of the field operators $\psi_i(C_i), i=1,\ldots,n$, by exactly the same arguments as those explained in [32].

Next, (7.2) would imply that

$$R(\tau_i,\sigma_j\circ\pi)\; R(\tau_j,\pi) \;=\; R(\tau_j,\sigma_i\circ\pi)\; R(\tau_i,\pi), \tag{7.14}$$

if $|\, i-j \,|\geq 2$, for every permutation π. But this is obvious from (7.6) and (7.7). ■

Remark. The representation R of B_n^c on $\mathcal{H}$ defined in (7.12) is equivalent to a unitary representation r of B_n^c on the vacuum sector $\mathcal{H}_0$ defined by

$$\begin{aligned} r(\tau_i^{\pm 1},\pi) &\equiv T^{-1}_{\mathring{\rho}\pi(n)}\cdots T^{-1}_{\mathring{\rho}\pi(1)}\; R(\tau_i^{\pm1},\pi)\; T_{\mathring{\rho}\pi(1)}\cdots T_{\mathring{\rho}\pi(n)}\\ &= T^{-1}_{\mathring{\rho}\pi(n)}\cdots T^{-1}_{\mathring{\rho}\pi(i+2)}\; \mathring{\varepsilon}^{\gtrless}_{\rho\pi(i)\mathring{\rho}\pi(i+1)}\; T_{\mathring{\rho}\pi(i+2)}\cdots T_{\mathring{\rho}\pi(n)}\\ &= \mathring{\rho}^{C_a}_{\pi(n)}\circ\cdots\circ\mathring{\rho}^{C_a}_{\pi(i+2)}\,(\mathring{\varepsilon}^{\gtrless}_{\rho\pi(i)\mathring{\rho}\pi(i+2)}), \end{aligned} \tag{7.15}$$

and the last equality in (7.15) follows from (5.15). For b as in (7.8), we set

$$\begin{aligned} r(b;\mathring{\rho}_1,\ldots,\mathring{\rho}_n) &= \prod_{m=k}^{1} r(\tau_{i_m}^{\epsilon_m},\,\pi_{m-1})\\ &= T^{-1}_{\mathring{\rho}_n}\cdots T^{-1}_{\mathring{\rho}_1}\; R(b;\mathring{\rho}_1,\ldots\mathring{\rho}_n)\; T_{\mathring{\rho}_1}\cdots T_{\mathring{\rho}_n}, \end{aligned} \tag{7.16}$$

and in the second equation we have used that $T_{\mathring{\rho}_i}$ and $T_{\mathring{\rho}_j}$ commute, for all i and j, by convention (6.36).

Corollary 7.2.

If $\mathring{\rho}_1=\cdots=\mathring{\rho}_n=\rho, [\rho]\in\mathcal{L}_{\pi_0}$, then

$$r(\tau_i^{\pm 1},\pi)\equiv r_\rho(\tau_i^{\pm1}) = (\rho^{C_a})^{\circ(n-i-2)}(\varepsilon_\rho^{\gtrless}) \tag{7.17}$$

is independent of π and

$$B_n\ni b \;\longrightarrow\; r_\rho(b) \;\equiv\; r(b;\rho,\ldots,\rho)$$

defines a unitary representation of B_n on the vacuum sector $\mathcal{H}_0$ which commutes with all operators in $(\rho^{C_a})^{\circ n}(\mathcal{B}^{C_a})$.

Proof. The only statement left to prove is that r commutes with $(\rho^{C_a})^{\circ n}(\mathcal{B}^{C_a})$: By part (2) of Theorem 6.4, $\varepsilon_\rho^{\gtrless}$ commutes with $\rho^{C_a}(\rho^{C_a}(\mathcal{B}^{C_a}))$. Hence (7.15),(7.16) show that $r(\tau_i^{\pm1})$ commutes with $(\rho^{C_a})^{\circ(n-i)}(\mathcal{B}^{C_a})$. Since

$$(\rho^{C_a})^{\circ k}(\mathcal{B}^{C_a}) \;\supseteq\; (\rho^{C_a})^{\circ n}(\mathcal{B}^{C_a}),$$

for all $k\neq n$, $r(\tau_i^{\pm1})$ commutes with $(\rho^{C_a})^{\circ n}(\mathcal{B}^{C_a})$, for all $i\leq n-1$. ■

Next, we investigate an important special situation, the one where ρ is an automorphism of $\mathcal{B}^{C_a}$. [A *morphism ρ of $\mathcal{B}^{C_a}$ is an automorphism if $\rho(\mathcal{B}^{C_a})=\mathcal{B}^{C_a}$. Since ρ is an isometry on $\mathcal{B}^{C_a}$, it follows that ρ^{-1} exists and is an automorphism of $\mathcal{B}^{C_a}$, as well.]

Theorem 7.3.

The following statements are equivalent:

(1) ρ^{C_a} is an automorphism of $\mathcal{B}^{C_a}$;

(2) $(\rho^{C_a})^2(\mathcal{B}^{C_a})' = \{\lambda\, \mathbb{I} : \lambda \in \mathbb{C}\}$, i.e., $(\rho^{C_a})^2$ is irreducible; and

(3) $\varepsilon_\rho^{\gtrless} = e^{\mp 2\pi i \theta(\rho)}\, \mathbb{I}$, for some $\theta(\rho) \in [0,1)$.

Proof. The proof is the same as the one of Proposition 2.7 in [7]. For the convenience of the reader we sketch it here: Clearly (1) implies (2). It has been shown in part (2) of Theorem 6.4 that $\varepsilon_\rho^{\gtrless}$ commutes with $\rho^{C_a}(\rho^{C_a}(\mathcal{B}^{C_a}))$. Hence (2) implies that

$$\varepsilon_\rho^{\gtrless} = \lambda_\rho^{\gtrless} \cdot \mathbb{I}.$$

Since $\varepsilon_\rho^{\gtrless}$ are unitary, $|\lambda_\rho^{\gtrless}| = 1$. By part (3) of Theorem 6.4, $\varepsilon_\rho^{>} \cdot \varepsilon_\rho^{<} = \mathbb{I}$, hence $\lambda_\rho^{>} \cdot \lambda_\rho^{<} = 1$, or $\lambda_\rho^{\gtrless} = e^{\mp 2\pi i \theta(\rho)}$, for some $\theta(\rho) \in [0,1)$.

It remains to show that (3) implies (1). We choose $\rho_1 = \rho, \Gamma_1 = \mathbb{I}, \rho_2 = \bar{\rho}, \Gamma_2 = \bar{\Gamma}$, where $\bar{\rho}^{C_a}(A) = \bar{\Gamma}\, \rho(A)\bar{\Gamma}^*$, for all $A \in \mathcal{B}^{C_a}, \bar{\rho}$ is localized in a space-like cone $\bar{\mathcal{C}}$, and e.g. as $(\rho) >$ as $(\bar{\rho})$, relative to $\mathcal{C}^a$. Then, by (6.9) and part (3),

$$\begin{aligned} \varepsilon_\rho^{>} &= \bar{\Gamma}_\rho^*\, \rho^{C_a}(\bar{\Gamma}) = e^{-2\pi i \theta(\rho)}, \text{ i.e.,} \\ \bar{\Gamma} &= e^{2\pi i \theta(\rho)}\, \rho^{C_a}(\bar{\Gamma}) \in \rho^{C_a}(\mathcal{B}^{C_a}). \end{aligned} \tag{7.18}$$

Now, let A be an operator in $\mathcal{B}^{C_a}$. By construction of $\mathcal{B}^{C_a}$, (see beginning of Sect. 5), given any $\varepsilon > 0$, then exists $B_\varepsilon \in \overline{\mathcal{A}(\mathcal{S})}^w$, for some simple domain $\mathcal{S} \times C_a + x$, for some $x \in \mathsf{M}^3$, such that $\|A - B_\varepsilon\| < \varepsilon$. We may choose the localization cone $\bar{\mathcal{C}}$ of $\bar{\rho}$ so that $\bar{\mathcal{C}} \times \mathcal{S}$ and $\bar{\mathcal{C}} \times C_a + y$, for some $y \in \mathsf{M}^3$. Then $\bar{\rho}^{C_a}(B_\varepsilon) = B_\varepsilon$, and (7.18) implies that

$$\begin{aligned} B_\varepsilon &= \varepsilon\bar{\rho}^{C_a}(B_\varepsilon) = \bar{\Gamma}\, \rho^{C_a}(B_\varepsilon)\bar{\Gamma}^* \\ &= \rho^{C_a}(\bar{\Gamma}\, B_\varepsilon\, \bar{\Gamma}^*) \in \rho^{C_a}(\mathcal{B}^{C_a}). \end{aligned}$$

This shows that $\mathcal{B}^{C_a} \subseteq \rho^{C_a}(\mathcal{B}^{C_a})$ and hence that ρ^{C_a} is an automorphism. ■

Remarks.

(1) Theorem 7.3 shows that, for automorphisms, ρ, of $\mathcal{B}^{C_a}$, the representation r_ρ of B_n is abelian, with $r_\rho(\tau_i^{\pm 1}) = e^{\mp 2\pi i \theta(\rho)}$. This is the situation realized in abelian gauge theories with anyons.

(2) If ρ is an automorphism then there exists a charge-conjugate automorphism $\bar{\rho} = \rho^{-1}$ such that $\bar{\rho}\circ\rho = \rho\circ\bar{\rho} = id.$. It is easy to see that $\theta(\bar{\rho}) = \theta(\rho)\ (mod\ \mathbb{Z})$, i.e. the charge-conjugate field has the same statistics as the original field.

Clearly, the case where ρ is an automorphism is special. If $(\rho^{C_a})^2(\mathcal{B}^{C_a})'$ is non-trivial then, a priori, we do not know more than that the representations r_ρ of B_n constructed in Corollary 7.2. are unitary. In contrast to the case of strictly local quantum theories studied by Doplicher, Haag and Roberts [7,26] which is understood completely, there is, in our case, only a rather rudimentary beginning of a general theory. We briefly sketch some results taken from [22,23,18] which are relevant in this context.

Given a morphism ρ^{C_a} of $\mathcal{B}^{C_a}$, we define a left inverse, ϕ, of ρ^{C_a} to be a linear map from $\mathcal{B}^{C_a}$ into $B(\mathcal{H}_0)$ with the properties:

(i) $\phi(A^*) = \phi(A)^*$, $\phi(A) \geq 0$ if $A \geq 0$;

(ii) $\phi(\mathbb{I}) = \mathbb{I}$;

(iii) $\phi(\rho^{C_a}(A)\, B\, \rho^{C_a}(C)) = A\,\phi(B)\,C$.

But, in general, $\phi(A \cdot B) \neq \phi(A) \cdot \phi(B)$, and $\phi(\mathcal{B}^{C_a}) \not\subseteq \mathcal{B}^{C_a}$, i.e., ϕ need not be a morphism. The principal results are as follows.

(a) Every morphism $\rho \in [\rho] \in \mathcal{L}_{\pi_0}$ has at least one left inverse [7].

(b) Suppose there exists a left inverse ϕ of ρ such that

$$\phi(\varepsilon_\rho^{>}) \neq 0. \tag{7.19}$$

Then there is a unique, so-called standard left inverse, ϕ_0, such that

$$\phi_0(\varepsilon_\rho^{>}) = \lambda_\rho\, U_\rho, \quad |\lambda_\rho| \leq 1,$$

where U_ρ is unitary, and $0 \neq \lambda_\rho \in \mathbb{C}$. Since $\varepsilon_\rho^>$ and $\varepsilon_\rho^<$ are unitary, with $\varepsilon_\rho^> \cdot \varepsilon_\rho^< = \mathbb{1}$, it follows that $\varepsilon_\rho^< = (\varepsilon_\rho^>)^*$ and hence by property (i) that

$$\phi_0(\varepsilon_\rho^<) = \bar{\lambda}_\rho U_\rho^*.$$

If ρ is irreducible, i.e., $\rho^{C_a}(\mathcal{B}^{C_a})' = \{\lambda \mathbb{1} : \lambda \in \mathbb{C}\}$, then $U_\rho = \mathbb{1}$. For, $\varepsilon_\rho^{\gtrless}$ commutes with $\rho^{C_a}(\rho^{C_a}(\mathcal{B}^{C_a}))$, hence $\phi(\varepsilon_\rho^{\gtrless})$ commutes with $\rho^{C_a}(\mathcal{B}^{C_a})$. If ρ is an automorphism then, by part (3) of Theorem 7.3,

$$\lambda_\rho = \phi(\varepsilon_\rho^>) = \rho^{-1}(\varepsilon_\rho^>) = e^{-2\pi i\theta(\rho)}. \tag{7.20}$$

The number λ_ρ is called the <u>statistics parameter</u> of ρ.

(c) The <u>statistical dimension</u>, $d(\rho)$, of ρ is defined as

$$d(\rho)^{-2} = |\lambda_\rho|^2 \leq 1. \tag{7.21}$$

If ρ_1 and ρ_2 are irreducible morphisms with $\lambda_{\rho_1} \neq 0 \neq \lambda_{\rho_2}$ then

$$d(\rho_1 \circ \rho_2) = d(\rho_1)\, d(\rho_2).$$

If $\rho = \bigoplus_{i=1}^{m} \rho_i$, where ρ_i is irreducible, for all i, then

$$d(\rho) = \sum_{i=1}^{m} d(\rho_i) \geq m. \tag{7.22}$$

(d) Under some additional assumptions (Poincaré covariance), see [8,37] and Sect. 8, one can show that if $\phi(\varepsilon_\rho^>) \neq 0$ then there exists a charge-conjugate morphism, $\bar{\rho}$, localized in some cone $\bar{C} \times C_a$ such that $\rho^{C_a} \circ \bar{\rho}^{C_a}$ and $\bar{\rho}^{C_a} \circ \rho^{C_a}$ contain the identity morphism precisely once [7,37], and

$$\lambda_\rho = \lambda_{\bar{\rho}}. \tag{7.23}$$

Let $\rho \in [\rho] \in \mathcal{L}_{\pi_0}$ be a direct sum of irreducible morphisms in $\mathcal{L}_{\pi_0}$, each of which has a charge-conjugate morphism defining a representation in $\mathcal{L}_{\pi_0}$. Then

$$d(\rho) \in \{2\cos\frac{\pi}{N} : N = 3,4,\ldots\} \cup [2,\infty]. \tag{7.24}$$

This is proven in [22]. Since $d(\rho) \geq 1$, for every morphism $\rho \in [\rho] \in \mathcal{L}_{\pi_0}$, it follows from (7.22) that if $d(\rho) \in I \equiv \{2\cos\frac{\pi}{N} : N = 3,4,\ldots\}$ then ρ is irreducible. In this case, $d(\rho) = d([\rho])$ only depends on the equivalence class of ρ. By comparing this situation with the one described in [7,8] it follows that if $d(\rho) \in I$ then the representation r_ρ of B_n does <u>not</u> reduce to a representation of the permutation group, S_n. [If ρ reduces to a representation of S_n then $d(\rho) = d([\rho]) \in \mathbb{N}$; see [7].]

(e) By taking over results from [22,18], one sees that if ρ is irreducible and $\rho^{C_a} \circ \rho^{C_a}$ has exactly two irreducible subrepresentations then

$$r_\rho(\tau_i^{\pm 1}) = e^{\mp 2\pi i\theta(\rho)}\, r_0(\tau_i^{\pm 1}), \tag{7.25}$$

for some $\theta(\rho) \in [0,1)$, where r_0 is either an infinite multiple of a representation of S_n with Young tableaux of $\leq d(\rho)$ rows or columns [7] or if $1 < d(\rho) < 2$ then $d(\rho) \in I$, and r_0 is an infinite multiple of the Ocneanu-Wenzl representation of B_n [23].

So far, there are no natural physical models in two space dimensions known in which the second case is realized, but it does appear in SU(2) pure Chern-Simons theory [13].

In the second case, the braid matrices $r_0(\tau_i^{\pm 1})$ can be expressed in terms of the R-matrices of the quantum group $U_q(sl(2))$, [25], with $q = e^{2\pi i/k}$. This quantum group then plays the role of a global, internal symmetry of the quantum theory. In analogy to deep results of Doplicher and Roberts [26] one might conjecture that, in theories of the type considered in this paper, and, for an irreducible morphism $\rho \in [\rho] \in \mathcal{L}_{\pi_0}$ with $d(\rho) < \infty$ which has a charge-conjugate morphism, the representation r_ρ of B_n is always of the form

$$r_\rho(\tau_i^{\pm 1}) = e^{\mp 2\pi i\theta(\rho)}\, r_0(\tau_i^{\pm 1}),$$

where r_0 is either a representation of S_n which comes from the representation theory of a compact internal symmetry group, or r_0 is a representation of B_n which comes from the representation theory of a quantum group, for all $n = 2, 3, \ldots$. Unfortunately, this is, at present, a speculation only verified in examples.

(f) In order to lend some support to the idea that there ought to be a general theory of the statistics of superselection sectors in three space-time dimensions, we wish to briefly discuss a useful element of a general theory, the so-called fusion rules [27,28,21]: Let $\{\rho_k\}_{k\in S}$ be a complete list of irreducible morphisms of $\mathcal{B}^{C_a}$ localized in space-like cones, as introduced in (4.1)-(4.4). By a "complete list", S, we mean that, for arbitrary i and j in S,

$$\rho_i^{C_a} \circ \rho_j^{C_a} = \bigoplus_{k\in I} \bigoplus_{\alpha=1}^{N_{kij}} \rho_{k(\alpha)}^{C_a}, \tag{7.26}$$

where the representation $\rho_{k(\alpha)}^{C_a}$ of $\mathcal{A}$ on $\mathcal{H}_0$ is unitary equivalent to the representation $\rho_k^{C_a}$ of $\mathcal{A}$ on $\mathcal{H}_0$, for all $\alpha = 1, \ldots, N_{kij}, k \in S$. The non-negative integer N_{kij} is the multiplicity of the representation $\rho_k^{C_a}$ in $\rho_i^{C_a} \circ \rho_j^{C_a}$. By Proposition 5.4, $\rho_i^{C_a} \circ \rho_j^{C_a}$ and $\rho_j^{C_a} \circ \rho_i^{C_a}$ are unitary equivalent. Hence

$$N_{kij} = N_{kji}. \tag{7.27}$$

We shall assume that there is an involution, $^-$, on S, with $^- = k \in S \to \bar{k} \in S$, such that, for every $k \in I$, $\rho_{\bar{p}}^{C_a}$ is charge-conjugate to $\rho_k^{C_a}$, i.e. $\rho_k^{C_a} \circ \rho_{\bar{p}}^{C_a} \simeq \rho_{\bar{p}}^{C_a} \circ \rho_k^{C_a}$ contain the identity representation, $\rho_1^{C_a}$, precisely once. [Results in [8,37] show that if $d(\rho_k) < \infty$, for all $k \in S$, and the representations ρ_k are Poincaré-covariant, for all $k \in S$, see Sect. 8, then the involution $^-$ exists.] We may now interpret N_{kij} as the multiplicity of $\rho_1^{C_a}$ in $\rho_{\bar{k}}^{C_a} \circ \rho_i^{C_a} \circ \rho_j^{C_a}$. Using again Proposition 5.4, we see that

$$\rho_{\bar{k}}^{C_a} \circ \rho_i^{C_a} \circ \rho_j^{C_a} \simeq \rho_i^{C_a} \circ \rho_{\bar{k}}^{C_a} \circ \rho_j^{C_a} \simeq \rho_j^{C_a} \circ \rho_i^{C_a} \circ \rho_{\bar{k}}^{C_a}.$$

Hence

$$N_{kij} = N_{\bar{i}kj} = N_{\bar{j}ik}. \tag{7.28}$$

Next, we consider $\rho_i^{C_a} \circ \rho_j^{C_a} \circ \rho_k^{C_a}$. By (7.26)

$$\rho_j^{C_a} \circ \rho_k^{C_a} = \bigoplus_{m\in I} \bigoplus_{\alpha=1}^{N_{mjk}} \rho_{m(\alpha)}^{C_a}, \quad \text{and}$$

$$\rho_i^{C_a} \circ \rho_m^{C_a} = \bigoplus_{n\in I} \bigoplus_{\beta=1}^{N_{nim}} \rho_{n(\alpha)}^{C_a}.$$

Thus, the representation $\rho_n^{C_a}$ appears precisely

$$\sum_{m\in I} N_{nim}\, N_{mjk} \tag{7.29}$$

times in $\rho_i^{C_a} \circ \rho_j^{C_a} \circ \rho_k^{C_a}$. Using again Proposition 5.4, it follows that

$$\sum_{m\in I} N_{nim}\, N_{mjk} = \sum_{m\in I} N_{njm}\, N_{mik}. \tag{7.30}$$

We define an $| S | \times | S |$ matrix, N_i, with matrix elements in $\mathbb{Z}_+$ by setting

$$(\mathsf{N}_i)_{jk} = N_{jik}. \tag{7.31}$$

Then (7.30) says that

$$[\mathsf{N}_i, \mathsf{N}_j] = 0, \quad \text{for all } i, j \text{ in } S. \tag{7.32}$$

Since $\rho_1^{C_a}$ is the identity, $N_{k1j} = \delta_{kj}$, for all k, j in S, i.e.

$$\mathsf{N}_1 = \mathbb{1}. \tag{7.33}$$

Thus, with the representations $\{\rho_k^{C_a}\}_{k\in S}$ of $\mathcal{A}$ localized in space-like cones, one can canonically associate

(i) statistical dimensions, $d(\rho_k)$;

(ii) (unitary equivalence classes of) statistics operators, $\varepsilon_{\rho_k}^{\gtrless}, k \in I$, and $\varepsilon_{\rho_i,\rho_j}^{\gtrless}, i, j \in S$, which determine unitary representations of $B_n^{(c)}$ and $S_n^{(c)}$, for all $n = 2, 3, \ldots$; and

(iii) a family with $| S |$ elements of $| S | \times | S |$ matrices N_i with non-negative, integer matrix elements (multiplicities) satisfying (7.27),(7.28) and (7.33) which commute with each other. They are called fusion rules.

The problem is to show that (i) - (iii) imply that S can be interpreted as the set of finite-dimensional, irreducible highest-weight representations of a compact group, G, in which case $d(\rho_k) = 1, 2, 3, \ldots$, for all $k \in S$, [7], or of some quantum group, QG,

[25], in such a way that (i) - (iii) are consequences of the theory of tensor product representations of G, QG, respectively. This problem has been solved for theories in four or more dimensions in [26].

It is useful to consider some examples.

(1) If $S = \{1,2,\ldots,n\}$, $\bar{k} = n - k + 1, N_{ikj} = \delta_{i,k+j-1(mod\ n)}$, then the representations $\rho_k^{C_a}$ are *automorphisms, $d(\rho_k) = 1$, and $G = \mathbb{Z}_n$.

(2) Suppose that all morphisms $\rho_k^{C_a}, k \in S$, are self-conjugate, (i.e. $\bar{k} = k$, for all $k \in S$). Then, by (7.27) and (7.28), the matrices $\mathsf{N}_i, i \in S$, are symmetric. Let $|S|$ be finite. We define

$$s = \min_{\substack{i\in I \\ i\neq 1}} \|\mathsf{N}_i\|. \tag{7.34}$$

It follows from results in [29] that

$$s \in \{2 \cos\frac{\pi}{N} : N = 3,4,\ldots\} \cup [2,\infty].$$

Without loss of generality, we may assume that $s = \|\mathsf{N}_2\|$. Suppose that $s = 2 \cos\frac{\pi}{N}$, $N < \infty$. Then

$$(\mathsf{N}_2)_{ij} = N_{i2j} = \delta_{i,j\pm 1}, \tag{7.35}$$

and it follows from (7.27),(7.28),(7.32) and (7.33) that

$$N_{kij} = \begin{cases} 1, & \text{if } |i-1| \leq k \leq max(N-2, i+j), \\ 0, & \text{otherwise.} \end{cases} \tag{7.36}$$

This is shown, for example, in [28]. It is well known, see [25,29], that these are multiplicities of the quantum group $U_q(sl(2))$, with $q = e^{2\pi i/N}$. In this case, the representations r_{ρ_2} of $B_n, n = 2,3,\ldots$, are the ones described in (e) , above, with $(N-2)^2\,\theta(\rho) = 1 \bmod \mathbb{Z}$.

We see that in examples (1) and (2), a considerable amount of the mathematical structure of $\{\rho_k^{C_a}\}_{k\in S}$ is already coded into the fusion rules $\{\mathsf{N}_k\}_{k\in S}$. It would be interesting to know how general this observation is.

The themes discussed here, along with some applications to two-dimensional condensed matter physics, will be discussed in more detail elsewhere.

8. $\mathcal{P}_+^\uparrow$-Covariant, Localizable Representations, Spin and Statistics.

In this section, we elucidate the role played by the unitary representation, U_ρ, of the quantum mechanical Poincaré group $\tilde{\mathcal{P}}_+^\uparrow$ associated with localizable representations in $\mathcal{L}_{\pi_0}$ described by morphisms, $\rho = \rho^{C_a}$, of the algebra $\mathcal{B}^{C_a}$.

We recall that in Sect. 3 the notion of representations of $\mathcal{A}$ localizable in cones (see Definition 3.3) was motivated by the properties of the covariant, positive energy representations describing (massive) one-particle states, see Definition 3.1 and Theorem 3.2 In Sects. 4-7, Poincaré covariance was irrelevant, but now we wish to describe its implications.

<u>Definition 8.1.</u> Let $\rho \in [\rho] \in \mathcal{L}_{\pi_0}$ be a representation localizable in cones relative to the vacuum representation π_0. We say that ρ is a $\tilde{\mathcal{P}}_+^\uparrow$-covariant representation of $\mathcal{B}^{C_a}$ if there exists a strongly continuous representation U_ρ of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_0$ such that

$$\rho^{C_a}(\alpha_L(A)) = U_\rho(L)\,\rho^{C_a}(A)\,U_\rho(L)^*, \tag{8.1}$$

for all $L = (\Lambda, x) \in \tilde{\mathcal{P}}_+^\uparrow$, with $U_\rho(L)^* = U_\rho(L)^{-1} = U_\rho(L^{-1})$. We say that a $\tilde{\mathcal{P}}_+^\uparrow$-covariant representation ρ is a positive-energy representation iff the generators $(P_0, \vec{P})$ of $U_\rho(x) \equiv U_\rho((\mathbb{1}, x))$ satisfy the relativistic spectrum condition (3.12).

□

<u>Remark.</u> This definition is merely a transcription of Definition 3.1: If the representation π_0 of $\rho^{C_a}(\mathcal{B}^{C_a})$ is unitary equivalent to the representation $(\pi, \mathcal{H}_\pi)$ of $\mathcal{B}^{C_a}$, with $\pi \in \mathcal{L}_{\pi_0}$, then

$$U_\rho(L) = T_\rho^{-1}\,U_\pi(L)\,T_\rho, \tag{8.2}$$

where T_ρ is the isometry from $\mathcal{H}_0$ onto $\mathcal{H}_\pi \equiv \mathcal{H}_{[\rho]}$ constructed in (5.10)-(5.15).

The subset of $\mathcal{L}_{\pi_0}$ consisting of $\tilde{\mathcal{P}}_+^\uparrow$-covariant, positive-energy representations is denoted by $\mathcal{L}_{\pi_0}^{cov}$.

Given a morphism $\rho \in \mathcal{L}_C$ localized in a space-like cone C, we define

$$\rho_L \equiv \alpha_L \circ \rho \circ \alpha_L^{-1}. \tag{8.3}$$

It then follows from (3.9) that ρ_L is localized in $\mathcal{C}_L = \{x \in \mathsf{M}^3 : L^{-1}x \in \mathcal{C}\}$.

Next, we construct a unitary intertwiner $\Gamma(L)$ between ρ and ρ_L: If U_0 is the unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_0$ then

$$\alpha_L(A) \;=\; U_0(L)\, A\, U_0(L)^{-1}, \tag{8.4}$$

on $\mathcal{H}_0$. By (8.1)-(8.4),

$$\begin{aligned}\rho_L(A) &= \alpha_L(U_\rho(L)^{-1}\, \rho(A)\, U_\rho(L)) \\ &= U_0(L)\, U_\rho(L^{-1})\, \rho(A) (U_0(L)\, U_\rho(L^{-1}))^*,\end{aligned} \tag{8.5}$$

for $A \in \mathcal{A}$. Thus, we define the intertwiner $\Gamma_\rho(L)$ by

$$\Gamma_\rho(L) \;=\; U_0(L)\, U_\rho(L^{-1}), \tag{8.6}$$

and then

$$\rho_L(A) \;=\; \Gamma_\rho(L)\, \rho(A)\, \Gamma_\rho(L)^*, \;\; A \in \mathcal{A}. \tag{8.7}$$

Since ρ_L is localized in $\mathcal{C}_L$ and ρ in $\mathcal{C}$, it follows from Proposition 4.2 and (4.13) that

$$\Gamma_\rho(L) \;\in\; (\mathcal{A}(\mathcal{C}') \cap \mathcal{A}(\mathcal{C}_L'))' \;=\; \mathcal{B}(\mathcal{C} \cup \mathcal{C}_L), \tag{8.8}$$

where the algebras $\mathcal{B}(\mathcal{C})$ have been defined in (4.9). It follows easily from (8.6) that $\Gamma_\rho(L)$ satisfies the "cocycle identity"

$$\Gamma_\rho(L_1 \cdot L_2)^* \;=\; \Gamma_\rho(L_1)^*\, \alpha_{L_1}(\Gamma_\rho(L_2)^*) \tag{8.9}$$

and that $\Gamma_\rho(L)$ is strongly continuous in L.

It is easy to see that existence of unitary operators $\{\Gamma_\rho(L) : L \in \tilde{\mathcal{P}}_+^\uparrow\}$, strongly continuous in L and satisfying (8.7) and (8.9) is, in fact, equivalent to the $\tilde{\mathcal{P}}_+^\uparrow$-covariance of the representation ρ: The operators $U_\rho(L)$, defined by

$$U_\rho(L) \;=\; \Gamma_\rho(L)^*\, U_0(L) \tag{8.10}$$

from the desired unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$ on $\mathcal{H}_0$.

In order to be able to take over the theory of covariant representations developed in [7], we require suitable localization properties for the "Poincaré cocycles" $\Gamma_\rho(L)$, sharpening (8.8). Our assumption is a variant of assumption (C2) in Sect. 4; see (4.18), (4.19).

(C2') Let $\{L(t) : 0 \le t \le 1\}$ be some path in $\tilde{\mathcal{P}}_+^\uparrow$, with $L(0) = (\mathbb{1}, 0)$, $L(1) = (\Lambda, x)$, and let $\mathcal{C}$ be some space-like cone. We set

$$\mathcal{C}(L(t)) \;=\; \{x \in \mathsf{M}^3 :\; L(t)^{-1}\, x \in \mathcal{C}\}. \tag{8.11}$$

Let $\mathcal{S}$ be a simple domain containing $\bigcup_{0 \le t \le 1} \mathcal{C}(L(t))$. Then

$$\Gamma_\rho(L(t)) \;\in\; \overline{\mathcal{A}(\mathcal{S})}^w, \;\text{ for all } 0 \le t \le 1. \tag{8.12}$$

By assumption (C3), it then follows that, for every $L \in \tilde{\mathcal{P}}_+^\uparrow$,

$$\Gamma_\rho(L) \;\in\; \mathcal{B}(\mathcal{C} \cup \mathcal{C}_L) \,\cap\, \overline{\mathcal{A}(\mathcal{S}_L)}^w, \tag{8.13}$$

with

$$\mathcal{S}_L \;\equiv\; \bigcap_{\gamma \in \Pi_L} \Big(\bigcup_{\substack{0 \le t \le 1 \\ \{L(t)\} = \gamma}} \mathcal{C}(L(t)) \Big), \tag{8.14}$$

where Π_L is the set of all paths in $\tilde{\mathcal{P}}_+^\uparrow$ connecting the origin $(\mathbb{1}, 0) \in \tilde{\mathcal{P}}_+^\uparrow$ to $L \in \tilde{\mathcal{P}}_+^\uparrow$, and $\gamma = \{L(t) :\; 0 \le t \le 1\}$ is an element of Π_L.

Let $\mathcal{C}_a$ be an auxiliary cone in the space-like complement of the localization cone, $\mathcal{C}$, of ρ. Since $\mathcal{C}$ and $\mathcal{C}_a$ are open sets, it follows from (8.13) that there is an open neighborhood, $U(\mathcal{C}, \mathcal{C}_a)$, of $(\mathbb{1}, 0)$ in $\tilde{\mathcal{P}}_+^\uparrow$ such that

$$\Gamma_\rho(L) \;\in\; \mathcal{B}^{\mathcal{C}_a}, \;\text{ for all } L \in U(\mathcal{C}, \mathcal{C}_a). \tag{8.15}$$

Let U be an arbitrary open neighborhood of $(\mathbb{1}, 0)$. Then

$$\{\Gamma_\rho(L) :\; L \in U\} \text{ determines } \{\Gamma_\rho(L) :\; L \in \tilde{\mathcal{P}}_+^\uparrow\} \text{ uniquely}. \tag{8.16}$$

For, every $L \in \tilde{\mathcal{P}}_+^\uparrow$ can be written as a product

$$L \;=\; L_1 \dots L_n, \;\; L_i \in U, \;\text{ for } i = 1, \dots, n. \tag{8.17}$$

Let $L^{(i)} \equiv L_1 \ldots L_i$. Then, by (8.9), (8.10),

$$\begin{aligned} U_\rho(L) &= \prod_{i=1}^{n} U_\rho(L_i) \\ &= \Gamma_\rho(L_1)^* \prod_{i=1}^{n-1} \alpha_{L_i}\,(\Gamma_\rho(L_{i+1})^*)\; U_0(L). \end{aligned} \tag{8.18}$$

Thanks to (8.15)-(8.18), the theory of $\tilde{\mathcal{P}}_+^\uparrow$- covariant representations localized in cones developed in [7,8] can be taken over to our framework essentially without changes. It follows, for example, that if ρ_1 and ρ_2 are $\tilde{\mathcal{P}}_+^\uparrow$-covariant morphisms localized in space-like cones C_1 and C_2, respectively, with $(C_1 \cup C_2) \times C_a$, for some auxiliary cone C_a, then $\rho_1^{C_a} \circ \rho_2$ is a $\tilde{\mathcal{P}}_+^\uparrow$-covariant morphism of the extended algebra $\mathcal{B}^{C_a}$. The key idea of the proof [7,8] is to notice that if L is in an open neighborhood U of $(\mathbb{I}, 0) \in \tilde{\mathcal{P}}_+^\uparrow$ so small that $(C_1 \cup C_2)(L) \times C_a$ then

$$\Gamma_{\rho_1 \circ \rho_2}(L) = \Gamma_{\rho_1}(L)\, \rho_1^{C_a}(\Gamma_{\rho_2}(L)) \tag{8.19}$$

is a Poincaré cocycle for $\rho_1 \circ \rho_2$ so that

$$U_{\rho_1 \circ \rho_2}(L) = \Gamma_{\rho_1 \circ \rho_2}(L)^*\, U_0(L) \tag{8.20}$$

is the desired representation of $\tilde{\mathcal{P}}_+^\uparrow$ with the property that

$$\rho_1^{C_a} \circ \rho_2(\alpha_L(A)) = U_{\rho_1 \circ \rho_2}(L)\, \rho_1^{C_a} \circ \rho_2(A)\, U_{\rho_1 \circ \rho_2}(L)^*. \tag{8.21}$$

Note that, by (8.15), the r.h.s. of (8.19) is well defined for $L \in U$. By (8.16)-(8.18), $\Gamma_{\rho_1 \circ \rho_2}(L)$, $L \in \tilde{\mathcal{P}}_+^\uparrow$, and hence $U_{\rho_1 \circ \rho_2}$, are uniquely determined by $\{\Gamma_{\rho_1 \circ \rho_2}(L) : L \in U\}$.

More generally, let $\rho_1, \ldots, \rho_n$ be $\tilde{\mathcal{P}}_+^\uparrow$-covariant morphisms localized in space-like cones, $C_1, \ldots C_n$, respectively such that $(C_1 \cup \cdots \cup C_n) \times C_a$. Then $\rho_1^{C_a} \circ \ldots \circ \rho_n^{C_a}$ is a $\tilde{\mathcal{P}}_+^\uparrow$-covariant morhpism of L, and for L in a sufficiently small, open neighborhood of $(\mathbb{I}, 0) \in \tilde{\mathcal{P}}_+^\uparrow$,

$$\Gamma_{\rho_1 \circ \ldots \circ \rho_n}(L) = \Gamma_{\rho_1}(L) \prod_{i=1}^{n-1} \rho_1^{C_a} \circ \ldots \circ \rho_i^{C_a}(\Gamma_{\rho_{i+1}}(L)) \tag{8.22}$$

which, by (8.15), is well defined and, by (8.16)-(8.18), determines $U_{\rho_1 \circ \ldots \circ \rho_n}$ uniquely.

We define

$$\Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L) = \Gamma_{\rho_1 \circ \ldots \circ \rho_{i-1}}(L)^* \Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L). \tag{8.23}$$

It follows from (8.22) that, for $L \in U$,

$$\Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L) = \rho_1^{C_a} \circ \ldots \circ \rho_{i-1}^{C_a}(\Gamma_{\rho_i}(L)). \tag{8.24}$$

Hence, using (8.3),(8.7) and (8.23), we see that

$$\begin{aligned} &\rho_1^{C_a} \circ \ldots \circ (\rho_i^{C_a})_L \circ \ldots \circ \rho_n^{C_a}(A) \\ &\quad = \Gamma_{\rho_1 \circ \ldots \circ \rho_{i-1}}(L)^* (\rho_1^{C_a})_L \circ \ldots \circ (\rho_i^{C_a})_L \circ \ldots \circ \rho_n^{C_a}(A) \Gamma_{\rho_1 \circ \ldots \circ \rho_{i-1}}(L) \\ &\quad = \Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L)\, \rho_1^{C_a} \circ \ldots \circ \rho_i^{C_a} \circ \ldots \circ \rho_n^{C_a}(A) \Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L)^*, \end{aligned} \tag{8.25}$$

for all $L \in \mathcal{P}_+^\uparrow$; (for $L \in U$, (8.25) also follows directly from (8.7) and (8.24)).

It is easy to see that (8.23), the cocycle identity (8.9) and (8.10) imply that

$$\begin{aligned} \Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L_1 \cdot L_2) = {} & \Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L_1)\, U_{\rho_1 \circ \ldots \circ \rho_i}(L_1) \\ & \Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L_2)\, U_{\rho_1 \circ \ldots \circ \rho_i}(L_1)^*. \end{aligned} \tag{8.26}$$

Let π^c denote the canonical projection from the covering group, $\tilde{\mathcal{P}}_+^\uparrow$, of $\mathcal{P}_+^\uparrow$ to $\mathcal{P}_+^\uparrow$. Since the automorphism group $\{\alpha_L : L \in \mathcal{P}_+^\uparrow\}$ is a representation of $\mathcal{P}_+^\uparrow$, it follows from (8.3) that, for all $i = 1, \ldots, n$,

$$(\rho_1^{C_a})_L \circ \ldots \circ (\rho_i^{C_a})_L(A) = (\rho_1^{C_a})_{\pi^c L} \circ \ldots \circ (\rho_i^{C_a})_{\pi^c L}(A), \tag{8.27}$$

for all $A \in \mathcal{B}^{C_a}$ and all $L \in \tilde{\mathcal{P}}_+^\uparrow$. Hence

$$\Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L_1)^*\, \Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L_2) \in \rho_1^{C_a} \circ \ldots \circ \rho_i^{C_a}(\mathcal{A})', \tag{8.28}$$

whenever $\pi^c L_1 = \pi^c L_2$.

Since $\{\alpha_L\}$ is a representation of $\mathcal{P}_+^\uparrow$, it follows from (8.9) and (8.28) that

$$\{\Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L)^* : \pi^c L = (\mathbb{I}, 0)\} \tag{8.29}$$

is a representation of the fundamental group, $\pi_1(\mathcal{P}_+^\uparrow) = \pi_1(SO(2)) = \mathbb{Z}$, of $\mathcal{P}_+^\uparrow$ with values in $\rho_1^{C_a} \circ \ldots \circ \rho_i^{C_a}(\mathcal{A})'$. Similarly, one can show that

$$\{\Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(L) : \pi^c L = (\mathbb{I}, 0)\} \tag{8.30}$$

is a representation of $\pi_1(\mathcal{P}_+^\uparrow)$ with values in $\rho_1^{C_a} \circ \ldots \circ \rho_i^{C_a}(\mathcal{A})'$. Let us denote the element $L \in \tilde{\mathcal{P}}_+^\uparrow$, with $\pi^c L = (\mathbb{I}, 0)$, in the homotopy class corresponding to $l \in \mathbb{Z}$ by l; l is e.g. a space rotation through an angle $2\pi l$. We set

$$\Gamma_{\rho_1 \circ \ldots \circ \rho_n}(l_1, l_2, \ldots, l_n) = \prod_{i=1}^n \Gamma^{(i)}_{\rho_1 \circ \ldots \circ \rho_n}(l_i). \tag{8.31}$$

Next, we note that $\{U_0(L) : L \in \tilde{\mathcal{P}}_+^\uparrow\}$ is really a representation of $\mathcal{P}_+^\uparrow$. For, the vacuum $\Omega \in \mathcal{H}_0$ is invariant under $U_0(L)$, and $\{\alpha_L : L \in \mathcal{P}_+^\uparrow\}$ is a representation of $\mathcal{P}_+^\uparrow$ as a *automorphism group on $\mathcal{A}$, so that

$$U_0(L) A \Omega = \alpha_L(A) \Omega$$

only depends on $\pi^c L$. Since $\{A\Omega : A \in \mathcal{A}\}$ is dense in $\mathcal{H}_0$, this proves our claim. We conclude that

$$U_{\rho_1 \circ \ldots \circ \rho_i}(L) = \Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L)^*, \quad \text{for } \pi^c L = (\mathbb{I}, 0). \tag{8.32}$$

Suppose that $\rho_1 \circ \ldots \circ \rho_i$ is irreducible. Since $\Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L) \in \rho_1^{C_a} \circ \ldots \circ \rho_i^{C_a}(\mathcal{A})' = \{\lambda \mathbb{I} : \lambda \in \mathbb{C}\}$, it follows from (8.32) that

$$U_{\rho_1 \circ \ldots \circ \rho_i}(L_{2\pi}) = e^{-2\pi i s(\rho_1 \circ \ldots \circ \rho_i)}, \tag{8.33}$$

if $L_{2\pi}$ is a space rotation through an angle 2π, where $s(\rho_1 \circ \ldots \circ \rho_i) \in [0,1)$ is the spin *mod.* $\mathbb{Z}$ of the representation $\rho_1 \circ \ldots \circ \rho_i$. In particular, if ρ is an irreducible one-particle representation then $s(\rho)$ is the spin of the particle *mod.* $\mathbb{Z}$ described by ρ. More generally, every irreducible subrepresentation $\hat{\rho}$ of $\rho_1 \circ \ldots \circ \rho_i$ corresponds to an eigenvalue $e^{2\pi i s(\hat{\rho})}$ of $\Gamma_{\rho_1 \circ \ldots \circ \rho_i}(L_{2\pi})$, where $s(\hat{\rho})$ is the spin *mod.* $\mathbb{Z}$ of the representation $\hat{\rho}$.

Next, we wish to relate the spin, $s(\rho)$, of a representation ρ, (in case ρ is irreducible), to its statistics operator, $\varepsilon_\rho^{\gtrless}$, defined in (6.25), (6.23) and (6.9). For this purpose, it is useful to express $\varepsilon_\rho^{\gtrless}$, or, more generally, $\varepsilon_{\rho,\hat{\rho}}^{\gtrless}$, (see (6.42) and (6.38)), in terms of the Poincaré cocycles, $\Gamma(L)$. In fact the cocycles for the rotation subgroup $\mathsf{R} = \widetilde{SO}(2)$ of $\tilde{\mathcal{P}}_+^\uparrow$ suffice for our considerations. Thus, let ρ be a $\tilde{\mathcal{P}}_+^\uparrow$-covariant morphism localized in a cone C. Let L_θ denote a rotation through an angle $\theta \in \mathsf{R} = \widetilde{SO}(2)$, and define

$$C_\theta = \{x \in \mathsf{M}^3 : L_\theta^{-1} x \in C\}, \text{ and} \tag{8.34}$$

$$\Gamma_\rho(\theta) \equiv \Gamma_\rho(L_\theta). \tag{8.35}$$

Then, by (6.9), Definition 6.2 and (8.23)

$$\varepsilon_\rho^{\gtrless} = \Gamma^{(2)}_{\rho\circ\rho}(\theta_1)^* \, \Gamma_\rho(\theta_2)^* \, \Gamma_\rho(\theta_1) \, \Gamma^{(2)}_{\rho\circ\rho}(\theta_2) \tag{8.36}$$

if $\theta_1 \gtrless \theta_2$, with $C_{\theta_1} \times C_{\theta_2}$ and

$$C_\theta \times C_a, \quad \text{for all } \theta \in [0, \theta_1] \cup [0, \theta_2], \tag{8.37}$$

for some auxiliary cone C_a.

Setting $\theta_2 = 0$ and choosing $\theta_1 = \theta \in (-2\pi, 0)$ such that $C_\theta \times C$ and $\bigcup_{\varphi \in [\alpha, 0]} C_\varphi \times C_a$, we obtain

$$\varepsilon_\rho^{<} = \Gamma^{(2)}_{\rho\circ\rho}(\theta)^* \, \Gamma_\rho(\theta). \tag{8.38}$$

Since $2\pi + \theta > 0$ and $C_\theta \times C$, we may choose an auxiliary cone $C_{\hat{a}}$, with $C_{\hat{a}} \times \bigcup_{\varphi \in [0, 2\pi + \theta]} C_\varphi$, and appeal to part (1) of Theorem 6.3 to see that

$$\varepsilon_\rho^{>} = \Gamma^{(2)}_{\rho\circ\rho}(2\pi + \theta)^* \, \Gamma_\rho(2\pi + \theta). \tag{8.39}$$

By the cocycle identity (8.9),

$$\Gamma_\rho(2\pi + \theta) = \Gamma_\rho(\theta) \, \Gamma_\rho(2\pi), \tag{8.40}$$

since $\alpha_{L_{2\pi}}$ is the identity. Moreover, by (8.23) and (8.9),

$$\Gamma^{(2)}_{\rho\circ\rho}(2\pi+\theta)^* = \Gamma_{\rho\circ\rho}(2\pi)^*\,\Gamma_{\rho\circ\rho}(\theta)^*\,\Gamma_\rho(\theta)\,\Gamma_\rho(2\pi). \tag{8.41}$$

If ρ is irreducible,

$$\Gamma_\rho(2\pi) = U_\rho(2\pi)^* = e^{2\pi i s(\rho)}, \tag{8.42}$$

and we find that

$$\begin{aligned}\varepsilon^>_\rho &= \Gamma_{\rho\circ\rho}(2\pi)^*\,\Gamma_{\rho\circ\rho}(\theta)^*\,\Gamma_\rho(\theta)^2\,e^{4\pi i s(\rho)}\\ &= \Gamma_{\rho\circ\rho}(2\pi)^*\,\Gamma^{(2)}_{\rho\circ\rho}(\theta)^*\,\Gamma_\rho(\theta)\,e^{4\pi i s(\rho)}\\ &= \Gamma_{\rho\circ\rho}(2\pi)^*\,\varepsilon^<_\rho\,e^{4\pi i s(\rho)}.\end{aligned} \tag{8.43}$$

We also have that $\varepsilon^>_\rho \cdot \varepsilon^<_\rho = \mathbb{1}$, see part (3) of Theorem 6.3, and hence (8.43) implies that

$$(\varepsilon^<_\rho)^2 = \Gamma_{\rho\circ\rho}(2\pi)\,e^{-4\pi i s(\rho)}. \tag{8.44}$$

If ρ is not irreducible but is a direct sum of irreducible representations then it is still true that $U_\rho(2\pi) = \Gamma_\rho(2\pi)^*$ commutes with $\Gamma_\rho(\theta)$ and we find that

$$\varepsilon^>_\rho = \Gamma_{\rho\circ\rho}(2\pi)^*\,\varepsilon^<_\rho\,\Gamma_\rho(2\pi)^2. \tag{8.45}$$

If ρ^{C_a} is an automorphism of $\mathcal{B}^{C_a}$, or, equivalently, $\rho^{C_a}\circ\rho^{C_a}(\mathcal{B}^{C_a})$ is irreducible, then Theorem 7.3 tells us that

$$\varepsilon^{\gtrless}_\rho = e^{\mp 2\pi i\theta(\rho)}\,\mathbb{1},\quad \theta(\rho)\in[0,1),$$

and, since $\Gamma_{\rho\circ\rho}(2\pi)^* = U_{\rho\circ\rho}(2\pi)$ commutes with $\rho^{C_a}\circ\rho(\mathcal{A})$,

$$\Gamma_{\rho\circ\rho}(2\pi) = e^{2\pi i s(\rho\circ\rho)}, \tag{8.46}$$

where $s(\rho\circ\rho)$ is the spin of the representation $\rho^{C_a}\circ\rho$. Then we have from (8.44) that

$$e^{4\pi i\theta(\rho)} = e^{2\pi i s(\rho\circ\rho)}\,e^{-4\pi i s(\rho)},$$

i.e.

$$s(\rho\circ\rho) = 2[s(\rho)+\theta(\rho)]\ mod.\ \mathbb{Z}. \tag{8.47}$$

If ρ is irreducible and $\rho^{C_a}\circ\rho$ is not irreducible, but can be decomposed into a direct sum of irreducible representations,

$$\rho^{C_a}\circ\rho = \bigoplus_k \rho_k$$

then $\varepsilon^{\gtrless}_\rho$ and $\Gamma_{\rho\circ\rho}(2\pi)^* = U_{\rho\circ\rho}(2\pi)$ can be diagonalized simultaneously, and, on the subspace, $\mathcal{H}_k$, of $\mathcal{H}_0$ carrying the representation ρ_k, $\varepsilon^{\gtrless}_\rho$ and $\Gamma_{\rho\circ\rho}(2\pi)^*$ are diagonal, i.e.

$$\begin{aligned}\varepsilon^{\gtrless}_\rho\,|_{\mathcal{H}_k} &= e^{\mp 2\pi i\theta^k(\rho)}\,\mathbb{1}\,|_{\mathcal{H}_k}\\ \Gamma_{\rho\circ\rho}(2\pi)^*\,|_{\mathcal{H}_k} &= e^{-2\pi i s(\rho_k)}\,\mathbb{1}\,|_{\mathcal{H}_k},\end{aligned}$$

and (8.44) yields

$$s(\rho_k) = 2[s(\rho)+\theta^k(\rho)]\ mod.\ \mathbb{Z}. \tag{8.48}$$

Hence

$$\varepsilon^<_\rho\,|_{\mathcal{H}_k} = \pm\,e^{2\pi i((s(\rho_k)/2)-s(\rho))}. \tag{8.49}$$

Equs. (8.47) and (8.48) are <u>spin addition rules</u> which express the spins of "two-particle representations" in terms of the spin of the one-particle representation and the eigenvalues of the statistics operator.

Next, we consider two different irreducible representations ρ and $\bar\rho$ carrying spin $s(\rho)$ and $s(\bar\rho)$ respectively. Repeating the calculations from equ. (8.35) to (8.45) we find that

$$\varepsilon^>_{\rho,\bar\rho} = \Gamma_{\bar\rho\circ\rho}(2\pi)^*\,\varepsilon^<_{\rho,\bar\rho}\,e^{2\pi i(s(\rho)+s(\bar\rho))}. \tag{8.50}$$

By a particular choice of conventions ($as(\rho) > as(\bar\rho)$, see the last remark of sect. 6), we may set $\varepsilon^>_{\rho,\bar\rho} = \mathbb{1}$. If $\bar\rho\circ\rho$ is a direct sum of irreducible representations,

$$\bar\rho^{C_a}\circ\rho = \bigoplus_k \rho_k \tag{8.51}$$

then

$$\epsilon^{<}_{\rho,\bar\rho}\,|\mathcal{H}_k \;=\; e^{2\pi i \theta^k(\rho;\bar\rho)} \tag{8.52}$$

$$\Gamma_{\bar\rho\circ\rho}(2\pi)^*\,|\mathcal{H}_k \;=\; e^{-2\pi i s(\rho_k)} \tag{8.53}$$

as in the case of a single morphism. Replacing (8.52) and (8.53) in (8.50) we find

$$s(\rho_k) \;=\; [s(\rho) \,+\, s(\bar\rho) \,+\, \theta^k(\rho;\bar\rho)] \;mod.\; \mathbf{Z}. \tag{8.54}$$

If $\bar\rho \circ \rho$ is irreducible, then (8.54) reduces to

$$s(\bar\rho\circ\rho) \;=\; (s(\rho) \,+\, s(\bar\rho) \,+\, \theta(\rho;\bar\rho)) \;mod.\; \mathbf{Z} \tag{8.55}$$

and if ρ and $\bar\rho$ are unitarily equivalent it is easy to see that

$$\theta(\rho;\bar\rho) \;=\; \theta(\rho) \,+\, \theta(\bar\rho) \;=\; 2\theta(\rho) \;mod.\,\mathbf{Z}, \tag{8.56}$$

so equation (8.55) gives again (8.47).

Let $\bar\rho$ be the morphism conjugate to a localized morphism ρ. Then $s(\rho) = s(\bar\rho)$ and $\theta(\rho) = \theta(\bar\rho)$. Furthermore, $\bar\rho\circ\rho$ contains the identity morphism. From that one can deduce that

$$0 \;=\; 2\,s(\rho) \,-\, 2\,\theta(\rho) \;mod.\,\mathbf{Z}, \text{ i.e.}$$
$$s(\rho) \;=\; \theta(\rho) \;mod.\,\frac{1}{2}\mathbf{Z}. \tag{8.57}$$

In analogy with the four-dimensional spin-statistics theorem [7,30] and with the anyon model, equ. (2.9), we actually expect that

$$s(\rho) \;=\; \theta(\rho) \;mod.\,\mathbf{Z}. \tag{8.58}$$

9. Scattering theory

The construction of asymptotic free particle states given by Haag and Ruelle in the axiomatic formalism of quantum field theory has been extensively discussed in the literature (see e.g. [33] and references therein), as well as its adaption to the algebraic framework ([7,8]). Most relevant to us is the version of scattering theory developed by Buchholz and Fredenhagen in [8] for non-local charged fields. Since their results can be carried over to the present situation, essentially without change, we shall only sketch how they are obtained and outline their significance. We refer the reader to the cited works for details.

Let V_m^+ be the forward hyperboloid of mass m and $\rho \in \mathcal{L}_C$ a covariant, irreducible, massive one-particle representation localized in the reference cone C. We denote the joint spectrum of the generators of translations in the representation ρ by $\sum_\rho$. $\sum_\rho$ contains the isolated hyperboloid V_m^+; (see Def. 3.1). Given a positive time-like vector e (which characterizes the direction of time in an appropriate Lorentz frame), and a space-like cone $\tilde C$, one defines a subspace of the test function space $\mathcal{S}(\mathbf{R}^3)$ as follows.

<u>Definition 9.1.</u> f belongs to the space $\mathcal{L}_\rho(\tilde C; e) \subseteq \mathcal{S}(\mathbf{R}^3)$ iff the Fourier transform $\tilde f$ of f satisfies:

(1) $supp\ \tilde f$ is compact and $supp\ \tilde f \cap \sum_\rho \subset V_m^+$;

(2) if $p \in supp\ \tilde f$ then
$$\frac{p - (p\cdot e)\,e}{\varepsilon_e(p)} \;\subset\; int\,(\tilde C - a) \tag{9.1}$$
where a is the apex of $\tilde C$, $int\,(\tilde C - a)$ denotes the interior of the cone $\tilde C - a$ and $\varepsilon_e(p) = (m^2 + (p\cdot e)^2 - p^2)^{1/2}$ is the relativistic energy in the Lorentz frame determined by e.

Let $\psi_\rho(B)$ be a field interpolating between charged sectors $\mathcal{H}_{[\hat\rho]}$ and $\mathcal{H}_{[\hat\rho\circ\rho]}$ of $\mathcal{H}_{tot}$. We define an extension to such fields of the action α_L of the Poincaré group on $\mathcal{A}$ as follows:

$$\alpha_L(\psi_\rho(B)\Big|_{\mathcal{H}[\hat\rho]}) \;=\; U_{\hat\rho\circ\rho}(L)\,\psi_\rho(B)\Big|_{\mathcal{H}[\hat\rho]}\,U_{\hat\rho}(L)^{-1} \tag{9.2}$$

where it is understood that we restrict our attention to covariant sectors in $\mathcal{L}^{cov}_{\pi_0}$. On $\mathcal{L}_{\pi_0}$, a similar definition for the action α_x of the translation subgroup of $\tilde{P}^{\uparrow}_{+}$ is always defined.

We may now construct a single-particle state of mass m in the sector $\mathcal{H}_{[\rho]}$ from a test function f in $\mathcal{L}_\rho(\tilde{C};e)$ and a field $\psi_\rho(B)$ in $\mathcal{F}^{C_a}(\tilde{C})$. This one-particle state will enter with certainty, at asymptotic times $t \to \infty$, the region $\tilde{C} + t\,e$ and is defined as follows:

$$\psi_\rho(f,t\,e) \;=\; \int d^3 x \int d^3 p\, e^{-ipx + i(p\cdot e - \epsilon_e(p))t}\, \tilde{f}(p)\, \alpha_x(\psi_\rho(B)), \tag{9.3}$$

the integral being understood in the weak sense. One verifies easily that $\psi_\rho(f,t\,e)\Omega$ is a single particle state which does not depend on t or e. The wellknown asymptotic behavior of smooth solutions of the Klein-Gordon equation [38, 39, 40] implies that $\psi_\rho(f,t\,e)$ is essentially localized in the region $\tilde{C} + t(e+s) \subseteq \tilde{C} + t\,e$ for some fixed vector s in $\tilde{C} - a$, a being the apex of $\tilde{C}$.

One can then prove for a set of irreducible morphisms $\rho_i \in \mathcal{L}_C$ and test functions $f_i \in \mathcal{L}_{\rho_i}(C_i;e), i = 1,\dots n$ the existence of the following strong limit in $\mathcal{H}_{[\rho_n \dots \rho_1]}$:

$$s - \lim_{t\to\infty} \psi_{\rho_1}(f_1;t\,e) \cdots \psi_{\rho_n}(f_n,t\,e)\,\Omega =: \psi^{out} \tag{9.4}$$

if the cones C_i are chosen mutually space-like separated and if $C'_1 \cap \dots \cap C'_n$ contains some auxiliary space-like cone C_a. Furthermore, if the localization cones $C_1,\dots C_n$ are kept fixed, ψ^{out} does not depend on the choice of the operators $\psi_{\rho_i}(f_i,t\,e)$ and of the Lorentz frame specified by e. Thus we may write

$$\psi^{out} \;=\; \psi_1 \overset{out}{\times} \dots \overset{out}{\times} \psi_n \tag{9.5}$$

and the closed linear span of the set of such vectors in $\mathcal{H}_{[\rho_n \circ \dots \circ \rho_1]}$ will be denoted by $\mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}(C_1,\dots,C_n;e)$.

It is also easy to derive the following behavior of $\psi^{out} \in \mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}(C_1,\dots C_n;e)$ under Poincaré transformations and braidings:

$$U_{\rho_n \dots \rho_1}(L)\,\psi^{out} \;=\; (U_{\rho_1}(L)\psi_1) \overset{out}{\times} \dots \overset{out}{\times} (U_{\rho_n}(L)\psi_n) \tag{9.6}$$

$$R(b;\rho_1,\dots,\rho_n)\,\psi^{out} \;=\; \psi_{\pi^{-1}(n)} \overset{out}{\times} \dots \overset{out}{\times} \psi_{\pi^{-1}(1)} \tag{9.7}$$

where π is the canonical permutation associated with $b \in B_n$, and R is defined as in Section 7.[6] Clearly, $U_{\rho_n \dots \rho_1}(L)\,\psi^{out} \in \mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}(L\,C_1;\dots L\,C_n; L\,e)$ and $R(b;\rho_1,\dots,\rho_n)\;\psi^{out} \in \mathcal{H}_{[\rho_{\pi^{-1}(n)} \dots \rho_{\pi^{-1}(1)}]}(C_{\pi^{-1}(1)};\dots C_{\pi^{-1}(n)};,e)$. Equation (9.7) implies in the case $\rho_1 = \dots = \rho_n$ that the permutation properties of wave functions describing n identical asymptotic particles will be given by the statistics operators R defined in Section 7.

Next, one proceeds to calculate the scalar product between two vectors $\psi^{out} \in \mathcal{H}_{[\rho_n \circ \dots \circ \rho_1]}(C_1,\dots,C_n;\,e)$ and $\hat{\psi}^{out} \in \mathcal{H}_{[\hat{\rho}_n \circ \dots \circ \hat{\rho}_1]}(\hat{C}_1,\dots,\hat{C}_n;\,e)$ where $\rho_i, \hat{\rho}_j \in \mathcal{L}_C$, $i,j = 1,\dots n$. (This scalar product will vanish if $\rho_i \not\simeq \hat{\rho}_i$ for some i, hence we may assume $\rho_i = \hat{\rho}_i$, $i = 1,\dots n$). The essential conclusion drawn from this computation (see [7, prop. 7.4] and [8, th. 8.3]) is that $\mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}(C_1,\dots,C_n;\,e)$ is isomorphic to (a subspace of) the tensor product of one particle spaces $\mathcal{H}^{out}_{[\rho_1]}(C_1;\,e) \otimes \dots \otimes \mathcal{H}^{out}_{[\rho_n]}(C_n;\,e)$ presenting n freely moving particles at asymptotic time. It is clear that $\mathcal{H}^{out}_{[\rho]}(C;\,e)$ is the subspace of one-particle states ψ whose energy-momentum is restricted by the condition

$$\{p - (p\,e)\cdot e \;\;|\, p \in\; supp\;\psi\} \subset\; C - a, \tag{9.8}$$

e is the apex of C.

Although the result found in [7, 8] is somewhat modified in the present context, the above conclusion still holds unchanged, thus completing the identification of outgoing states with freely moving particles at asymptotic times. A precise derivation of the scalar product of two asymptotic vectors $\psi_{out}, \hat{\psi}_{out}$ will be given in a more convenient setting elsewhere.

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[6] *In Sects. 6 and 7, the operators R were defined for reference morphisms $\rho_1,\dots\rho_n$ localized in space-like separated cones. This was convenient, since then the isometries $T_{\rho_i}, T_{\rho_j}; i \neq j$, may be chosen to commute. In the present case one has simply to keep track of the exact order of the $T_{\rho_i}, i = 1,\dots n$ appearing in $R(b;\rho_1,\dots,\rho_n)$.*

Finally, the closed linear span $\mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}$ of all the vectors contained in some $\mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}(C_1, \dots, C_n; e)$ is the space of outgoing n-particle scattering states which are composed of single particle states from the representations $\rho_1, \dots, \rho_n$. It can be shown that this construction is independent of the choice of the time-like vector e, that is of a particular Lorentz system. In the case of automorphisms $\rho_1 = \dots = \rho_n$ the structure of $\mathcal{H}^{out}_{[\rho_n \circ \dots \circ \rho_1]}$ has been carefully analysed in [5].

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