BRAID STATISTICS IN LOCAL QUANTUM THEORY

J. FRÖHLICH and F. GABBIANI

Theoretical Physics, ETH-Hönggerberg, CH-8093 Zürich

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We present details of a mathematical theory of superselection sectors and their statistics in local quantum theory over (two- and) three-dimensional space-time. The framework for our analysis is algebraic quantum field theory. Statistics of superselection sectors in three-dimensional local quantum theory with charges not localizable in bounded space-time regions and in two-dimensional chiral theories is described in terms of unitary representations of the braid groups generated by certain Yang-Baxter matrices. We describe the beginnings of a systematic classification of those representations. Our analysis makes contact with the classification theory of subfactors initiated by Jones. We prove a general theorem on the connection between spin and statistics in theories with braid statistics. We also show that every theory with braid statistics gives rise to a "Verlinde algebra". It determines a projective representation of $SL(2, \mathbb{Z})$ and, presumably, of the mapping class group of any Riemann surface, even if the theory does not display conformal symmetry.

1. Introduction

Local quantum field theory in two and three space-time dimensions has structural properties quite different from those of theories in four or more space-time dimensions. In three-dimensional theories with charges not localizable in bounded space-time regions, such as Chern-Simons gauge theory with charged matter fields, and in two-dimensional chiral theories, the main novel features are the following. The statistics of "charged fields" (more precisely, the statistics of superselection sectors) is described by certain unitary representations of the braid groups generated by Yang-Baxter matrices, so called *statistics matrices*, rather than by representations of the permutation groups, as is the case for local quantum theories in four or more dimensions. This feature has been anticipated in studies of particle statistics in the quantum mechanics of systems in two-dimensional space [1, 2], albeit in a less precise form. The braid statistics in two-dimensional systems is more than a theoretical curiosity. It has found important applications in the theoretical description of the fractional quantum Hall effect, due to Laughlin and others [3], and it may be an important ingredient in models of high-temperature superconductivity [4, 58]. These developments are briefly reviewed in [5, 24]. The results in this paper and in [24] provide a general theoretical foundation for braid statistics and its relevance in a theoretical description of two-dimensional systems with infinitely many degrees of freedom satisfying – at least approximately – locality (Einstein causality). It has been shown in [24] that the "charged fields" of a local quantum theory in two spatial dimensions exhibit braid statistics *only* if (i) they cannot be localized in bounded space-time regions (heuristically, this means that the

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theory has manifest or hidden local gauge invariance), if (ii) the corresponding charged states have fractional spin $\notin \frac{1}{2}\mathbb{Z}$, and if (iii), under a certain "minimality" assumption on the structure of superselection sectors, the discrete symmetries of space reflections in lines and time reversal are broken. Thus, two-dimensional systems with infinitely many degrees of freedom in a strong external magnetic field perpendicular to the plane of the system, as used in studies of the quantum Hall effect, or two-dimensional systems exhibiting a "flux phase" are natural candidates of systems exhibiting charged excitations with braid statistics. Among field theoretic models in three space-time dimensions with charged fields obeying braid statistics are *dynamical Chern-Simons gauge theories* (abelian and non-abelian) with charged matter fields, as discussed in [24] on the basis of results in [6, 7, 45, 68, 69, 70], and non-linear $0(3)-\sigma-$models with a Hopf term [8] which are essentially equivalent to certain abelian Chern-Simons-Higgs models.

The braid statistics of chiral vertices in two-dimensional conformal field theory is a fundamental structural property of these theories and plays an important role in the classification of rational theories. It has been studied in many recent papers [9, 10, 26, 41, 42, 43, 49, 66] in a variety of different formulations. As two-dimensional conformal field theories are basic building blocks for four-dimensional string theories, the analysis of braid statistics is likely to have significant applications in string theory, as already suggested through the work of Gervais and Neveu [11].

The ideas, results and techniques presented in this paper can be applied to the study of chiral sectors of two-dimensional conformal field theory, provided one starts from an algebraic formulation of these theories, as proposed by Buchholz, Mack and Todorov [12] and others [13, 22, 26]. There is, in fact, an intriguing, close connection between general three-dimensional local quantum theory and chiral sectors of two-dimensional conformal field theory that we shall discuss in Sec. 6. It is a general version of the connection between three-dimensional, topological Chern-Simons gauge theory and the chiral sectors of some two-dimensional conformal field theories found by Witten [69].

It might be clear from these remarks that we expect the braid statistics of low-dimensional local quantum theory to be more than an interesting mathematical structure. It is an intrinsic feature of some important systems in two-dimensional condensed matter physics, of the theory of critical phenomena in two dimensions, and, perhaps, of realistic string theories. This is one of the motivations underlying our work.

We would like to emphasize another basic structural property of local quantum theories with braid statistics. *Independently of conformal invariance*, a local quantum theory with braid statistics gives rise to a "Verlinde algebra"; its fusion rules are diagonalized by a unitary matrix, S, that can be expressed in terms of the *statistics matrices* of the theory. Furthermore, one can construct a unitary matrix T that can be expressed in terms of the *spins* of the superselection sectors of the theory and that depends on a real constant c, defined mod. 8, which, in conformal field theory, is interpreted as the *central charge* of the Virasoro algebra, but which, in the general context of theories with braid statistics, emerges as a basic invariant of the structure of sectors. One then shows that S and T determine a projective representation of $SL(2, \mathbb{Z})$. The idea that, in a theory where *all* charged sectors have non-trivial braid statistics, the S-matrix can be expressed in terms of the statistics matrices of the theory goes back to [41, 55]. In a context similar to the one of our paper, a construction of the matrices S and T has also been described, independently of our work, by Rehren [62].

It appears that these results can be generalized considerably. Using ideas from Moore and Seiberg [41], one expects to be able to construct projective representations of the mapping class groups of Riemann surfaces of arbitrary genus from the statistical data, i.e., from the fusion rules and the statistics matrices, of local quantum theories in which all sectors have non-trivial braid statistics. If correct, this conjecture will be quite significant in the theory of subfactors and in three-dimensional topology. It might be mentioned here that it has already been shown in [24, 55] how to construct invariants for links imbedded in S^3 from the statistical data of local quantum theories with braid statistics.

Thus we expect that the ideas and results discussed in this paper are not only making contact with some important problems in two-dimensional theoretical physics but also with some important topics in pure mathematics.

To conclude this introduction, we briefly summarize the contents of this paper.

In **Sec. 2**, we recall some basic aspects of the algebraic formulation of local quantum theory of systems with infinitely many degrees of freedom. We introduce algebras of local observables and develop their representation theory. We specify a class of "representations of interest", and we define a composition of representations in this class and the conjugation of a representation. These notions are analogous to the notions of the tensor product of two representations and conjugate representations in the representation theory of compact groups. They define a tensor category.

We then define fusion rules as multiplicities of irreducible subrepresentations of the composition of two irreducible representations. We also introduce Hilbert spaces of intertwiners that are analogues of the Clebsch-Gordan operators in group theory.

Of course, our formulation is by and large taken from the basic papers of Doplicher, Haag and Roberts [17, 18] and of Buchholz and Fredenhagen [20]. In particular, we use the fundamental result in [20] that "representations of interest" of the observable algebra are obtained by composing its groundstate, or vacuum representation with *-endomorphisms localized in space-like cones.

In **Sec. 3**, we construct vector bundles over spaces of *-endomorphisms of the observable algebra localized in space-like cones whose fibres are Hilbert spaces of intertwiners. We specify the transition functions of these bundles.

In **Sec. 4**, we construct different local sections of orthonormal frames of the intertwiner bundles and calculate the unitary transformations of the fibres mapping one section of orthonormal frames onto another one. These unitary transformations provide the *braid* (or statistics) and *fusion matrices* of the theory. Our construction is analogous to the construction of chiral vertices, braid and fusion matrices in two-dimensional conformal field theory.

We then derive the basic properties of the braid and fusion matrices and the polynomial equations between them. As an important corollary we establish a precise *connection between spin and statistics* in theories with braid statistics.

In **Sec. 5**, we discuss further properties of the intertwiner bundles and introduce the statistics parameter of an irreducible representation of the observable algebra, i.e., of a superselection sector [17,22]. The statistics parameter is an invariant of the statistics of a superselection sector. It is computed in terms of the braid and fusion matrices of the theory, and its properties are elucidated.

In **Sec. 6**, we analyze the connections between the theory of superselection sectors and their statistics in local quantum theory and the theory of subfactors of von Neumann algebras, as developed by Jones, Ocneanu, Pimsner, Popa, Wenzl and others [14,28,30,31,37,51,59,61]. Using results of Wenzl [59,64], we completely describe the braid statistics of certain fairly simple classes of local quantum theories. It is likely that three-dimensional Chern-Simons gauge theories with gauge groups $SU(n)$ and $SO(n)$ and two-dimensional Wess-Zumino-Novikov-Witten models are examples of such theories. Our results slightly extend prior results of Fredenhagen, Rehren and Schroer [22].

Finally, we show how one can construct two unitary matrices, S and T, from the braid and fusion matrices of a theory with sectors having non-trivial braid statistics, with the following properties: S diagonalizes the fusion rules of the theory, and $TSTST = S$.

This paper is a compromise between a research and a review paper. Many results have been worked out independently of other peoples' work. We believe that certain aspects of our approach and some of our results are new. But we wish to acknowledge that there is a very sizable overlap between this work and the work of Fredenhagen, Rehren and Schoer [22,62] and of Longo [30,31]. Since a considerable part of our work, as already described in [23,24,26] has been developed independently, and since it is sometimes useful to study complicated problems from several different points of view, we believe it still makes sense to present the details of our approach and of our arguments.

Of course, our work relies in a fundamental way on the ideas developed in [17,18,20].

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2. The Algebraic Approach to Quantum Field Theory

The basic objects of the algebraic approach to quantum field theory [15,16] are those physical observables of a system that can be measured in some bounded region of space-time: to each open region $\mathcal{O} \subseteq \mathbb{M}^3$, we associate a von Neumann algebra $\mathcal{A}(\mathcal{O})$

of observables. These algebras are required to have the following properties, reflecting physical principles which are believed to hold in every local, relativistic quantum field theory:

(i) Isotony: if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$.

(ii) Locality: if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated regions (this will be denoted by $\mathcal{O}_1 \not\wr \mathcal{O}_2$) then $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute, i.e.

$$[A; B] = 0, \qquad \forall A \in \mathcal{A}(\mathcal{O}_1), \qquad B \in \mathcal{A}(\mathcal{O}_2). \tag{2.1}$$

We define the "algebra of all quasi-local observables" as

$$\mathcal{A} := \overline{\bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})}^{n}, \tag{2.2}$$

where the closure is taken in the norm.

(iii) Poincaré covariance: there exists a representation of the Poincaré group $P_+^\uparrow$ as a group of $*$-automorphisms of $\mathcal{A}$, $\alpha_{(\Lambda,a)} : \mathcal{A} \to \mathcal{A}$, $(\Lambda, a) \in P_+^\uparrow$, such that

$$\alpha_{(\Lambda,a)}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}_{(\Lambda,a)}) \tag{2.3}$$

holds, where $\mathcal{O}_{(\Lambda,a)}$ is the image of $\mathcal{O}$ under the Poincaré transformation (Λ, a).

For the sake of simplicity, we assume, from now on, that the regions $\mathcal{O}$ are open double cones, i.e. the intersection of a forward light cone with a backward light cone. We also consider infinitely extended regions, such as $\mathcal{O}'$ (the spacelike complement of a double cone $\mathcal{O}$), or *spacelike cones* which are defined as follows: let $\mathcal{O}$ be an open double cone whose closure $\bar{\mathcal{O}}$ lies in the spacelike complement of the origin in $\mathbb{M}^3$. Then

$$\mathcal{C} = a + \bigcup_{\lambda > 0} \lambda \cdot \mathcal{O}, \qquad a \in \mathbb{M}^3 \tag{2.4}$$

is a spacelike cone with apex a. For such infinitely extended regions, we define,

$$\begin{aligned} \mathcal{A}(\mathcal{O}') &:= \overline{\bigcup_{\mathcal{O}_1 \subseteq \mathcal{O}'} \mathcal{A}(\mathcal{O}_1)}^{n}, \\ \mathcal{A}(\mathcal{C}) &:= \overline{\bigcup_{\mathcal{O}_1 \subseteq \mathcal{C}} \mathcal{A}(\mathcal{O}_1)}^{n}. \end{aligned} \tag{2.5}$$

The relative commutant $\mathcal{A}^c(\mathcal{C})$ of the algebra $\mathcal{A}(\mathcal{C})$ for some region $\mathcal{C}$ is defined as the set of observables in $\mathcal{A}$ which commute with $\mathcal{A}(\mathcal{C})$.

The properties of a physical system described by a pair $\{\mathcal{A}, \alpha\}$ can be inferred from the representation theory of $\{\mathcal{A}, \alpha\}$, provided some criterion which singles out the physically relevant representations is given. The superselection sectors of the system are then described by unitarily inequivalent, irreducible representations of $\mathcal{A}$.

In a series of basic papers [17, 18, 19] Doplicher, Haag and Roberts fully elucidated the superselection structure of theories based on the following selection criterion [17].

A representation j of $\mathscr{A}$ is considered as "interesting for particle physics" if

$$j(\mathscr{A}(\mathcal{O}')) \cong 1(\mathscr{A}(\mathcal{O}')) \tag{2.6}$$

holds for any bounded double cone $\mathcal{O}$, where 1 is the vacuum representation of $\mathscr{A}$, (i.e., an irreducible representation of $\mathscr{A}$ determined by a Poincaré-invariant state, ω, on $\mathscr{A}$), on the separable Hilbert space $\mathscr{H}_1$. The sign "$\cong$" in Eq. (2.6) means unitary equivalence. The superselection structure of such a physical system can be described in terms of the representation theory of some compact group which plays the role of a gauge group of the first kind for the system [19].

The selection criterion (2.6) singles out physical states which are indistinguishable from the vacuum state ω outside any bounded space-time region and automatically rules out gauge theories [17, 20].

In an alternative approach [20], Buchholz and Fredenhagen consider representations $(j, \mathscr{H}_j)$ of $\mathscr{A}$ on a Hilbert space $\mathscr{H}_j$ which satisfy the following selection criterion introduced by Borchers [46]. The Hilbert space $\mathscr{H}_j$ carries a strongly continuous, unitary representation of space-time translations implementing the corresponding automorphisms $\alpha_{(1,x)}$ of $\mathscr{A}$, i.e.,

$$U_j(x) j(A) U_j(x)^* = j(\alpha_{(1,x)}(A)). \tag{2.7}$$

Furthermore, the joint spectrum of the generators, P_j, of $U_j(x)$ satisfies

$$\operatorname{spec} P_j \subseteq \bar{V}^+,$$

where $\bar{V}^+$ denotes the closure of the forward light cone. This is the relativistic spectrum condition. Such representations are called *positive-energy representations.*

For systems which admit a complete particle interpretation and without zero-mass particles, results in [20] show that a positive-energy representation has the property that, for any spacelike cone $\mathscr{C}$,

$$j(\mathscr{A}^c(\mathscr{C})) \cong 1(\mathscr{A}^c(\mathscr{C})) \tag{2.8}$$

holds. Such representations are said to be *localizable in spacelike cones.* Condition (2.8) is weaker than (2.6) and does not exclude a priori gauge theories. There is even some evidence that it might hold in systems with zero-mass particles [21], for $\mathscr{C}$ fixed.

Buchholz and Fredenhagen then extend the analysis of [17, 18, 19] to the class of representations localizable in spacelike cones. This involves some technical arguments which will be sketched presently. A crucial ingredient of their analysis is the *duality assumption* for spacelike cones in the vacuum representation $(1, \mathscr{H}_1)$ [17, 20],

$$1(\mathscr{A}^c(\mathscr{C}))' = 1(\mathscr{A}(\mathscr{C}))^{-w}. \tag{2.9}$$

The symbols $'$ and $^{-w}$ mean "taking the commutant" and "closure in the weak operator topology of $\mathscr{B}(\mathscr{H}_1)$", respectively. We will require a somewhat modified version of this duality assumption. First, we restrict our attention to a particular kind of domain. Let $\mathscr{C}_0$ be a wedge in the two-dimensional $\{t = 0\}$-plane. The causal completion, $S := (\mathscr{C}_0')'$, of $\mathscr{C}_0$ is called a *simple domain*. If the opening angle of $\mathscr{C}_0$ is smaller than π, then $\mathscr{S}$ is a spacelike cone. We will assume that

$$1(\mathscr{A}^c(\mathscr{S}))' = 1(\mathscr{A}(\mathscr{S}))^{-w} \tag{2.10}$$

holds for simple domains in the vacuum representation.

If $(j, \mathscr{H}_j)$ is a representation localizable in cones, then Eq. (2.8) implies [20] the existence of an isometry $V_\mathscr{C}$ from $\mathscr{H}_j$ onto the vacuum sector $\mathscr{H}_1$ such that

$$j(A) = V_\mathscr{C}^* A V_\mathscr{C}, \qquad \forall A \in \mathscr{A}^c(\mathscr{C}). \tag{2.11}$$

(From now on, we drop the symbol $1(\cdot)$ and identify $\mathscr{A}$ with its vacuum representation $1(\mathscr{A})$.) The construction of such an isometry involves using the Reeh-Schlieder theorem and will not hold in cases where the Reeh-Schlieder theorem does not apply. We may then define a representation $\rho_\mathscr{C}^j \equiv \rho_\mathscr{C}$ of $\mathscr{A}$ on $\mathscr{H}_1$ which is unitarily equivalent to j:

$$\rho_\mathscr{C}(A) = V_\mathscr{C}\, j(A) V_\mathscr{C}^*. \tag{2.12}$$

For $A \in \mathscr{A}$, $\rho_\mathscr{C}(A)$ is a bounded operator on $\mathscr{H}_1$ and by (2.11) $\rho_\mathscr{C}$ acts trivially on $\mathscr{A}^c(\mathscr{C})$:

$$\rho_\mathscr{C}(A) = A, \qquad \forall A \in \mathscr{A}^c(\mathscr{C}). \tag{2.13}$$

We say that the representation $\rho_\mathscr{C}$ is *localized in the spacelike cone* $\mathscr{C}$.

Performing the construction (2.12) for two different cones $\mathscr{C}_1$ and $\mathscr{C}_2$, we get two unitarily equivalent representations $\rho_{\mathscr{C}_1} \equiv \rho_1$ and $\rho_{\mathscr{C}_2} \equiv \rho_2$ on $\mathscr{H}_1$, i.e.,

$$\rho_1(A)\Gamma_{\rho_1\rho_2} = \Gamma_{\rho_1\rho_2}\rho_2(A). \tag{2.14}$$

The unitary operators $\Gamma_{\rho_1\rho_2}$ are "*charge transport operators*" and play a basic role in the following. Using the localization property (2.13) of $\rho_\mathscr{C}$ and the duality assumption (2.10), one can show that

(i) $\rho_\mathscr{C}(\mathscr{A}(\mathscr{S})) \subseteq \mathscr{A}(\mathscr{S})^{-w}$, for every simple domain $\mathscr{S} \supseteq \mathscr{C}$.
(ii) $\Gamma_{\rho_1\rho_2} \in \mathscr{A}(\mathscr{S})^{-w}$ if $\mathscr{S}$ is some simple domain such that $\mathscr{C}_1 \cup \mathscr{C}_2 \subseteq \mathscr{S}$. (2.15)

Property (i) implies that the representation $\rho_\mathscr{C}$ of $\mathscr{A}$ is, in general, *not* given by a *-endomorphism of $\mathscr{A}$, and property (ii) implies that the charge transport operators $\Gamma_{\rho_1\rho_2}$ do *not* necessarily belong to $\mathscr{A}$. This rather impractical situation is improved by introducing an *auxiliary* C^*-algebra $\mathscr{B}^{\mathscr{C}_a}$ (see [19] for an alternative approach). Let $\mathscr{C}_a$ be some auxiliary cone of arbitrarily small opening angle and let

$$\mathscr{C}_a + x = \{y \in \mathbb{M}^3 : y - x \in \mathscr{C}_a\}. \tag{2.16}$$

We define the enlarged algebra $\mathscr{B}^{\mathscr{C}_a}$ containing $\mathscr{A}$ by setting

$$\mathscr{B}^{\mathscr{C}_a} = \left(\bigcup_{x \in \mathbb{M}^3} \overline{\mathscr{A}^c(\mathscr{C}_a + x)}^w \right)^{-n} . \tag{2.17}$$

It has been shown in [20] that $\rho_{\mathscr{C}}$ has a unique, continuous extension to $\mathscr{B}^{\mathscr{C}_a}$ which is weakly continuous on $\mathscr{A}^c(\mathscr{C}_a + x)^{-w}$, and coincides with $\rho_{\mathscr{C}}$ on $\mathscr{A}$. If $\mathscr{C}$ is spacelike separated from $\mathscr{C}_a + x$ for some $x \in \mathbb{M}^3$, then $\rho_{\mathscr{C}}$ is a **-endomorphism* of $\mathscr{B}^{\mathscr{C}_a}$, i.e., $\rho_{\mathscr{C}}$ is a linear map from $\mathscr{B}^{\mathscr{C}_a}$ into $\mathscr{B}^{\mathscr{C}_a}$ such that

$$\rho_{\mathscr{C}}(A \cdot B) = \rho_{\mathscr{C}}(A)\rho_{\mathscr{C}}(B) \qquad \text{and} \qquad \rho_{\mathscr{C}}(A^*) = \rho_{\mathscr{C}}(A)^* . \tag{2.18}$$

Moreover, locality implies that $\rho_{\mathscr{C}}$ is faithful and norm-preserving. If $\mathscr{S}$ is the spacelike complement of $\mathscr{C}_a + x$ for some x, then we see from (2.15) (ii) that the charge transport operators $\Gamma_{\rho_1\rho_2}$ belong to the auxiliary algebra $\mathscr{B}^{\mathscr{C}_a}$. Once we have chosen an auxiliary cone $\mathscr{C}_a$, we may extend the representations j of $\mathscr{A}$ to representations of $\mathscr{B}^{\mathscr{C}_a}$ as follows. Pick a representation ρ on the vacuum sector which is unitarily equivalent to j and is localized spacelike to $\mathscr{C}_a + x$, for some $x \in \mathbb{M}^3$. Extend the representation ρ to an endomorphism of $\mathscr{B}^{\mathscr{C}_a}$ and then use Eq. (2.12) to define the representation $j^{\mathscr{C}_a}$ of $\mathscr{B}^{\mathscr{C}_a}$. For an irreducible representation j, it is easy to check the following facts [20]:

(i) The representation $j^{\mathscr{C}_a}$ coincides with j on the subalgebra $\mathscr{A}$ of $\mathscr{B}^{\mathscr{C}_a}$.

(ii) The definition of $j^{\mathscr{C}_a}$ is independent of the particular choice of the representation ρ on the vacuum sector used in the construction, as long as ρ is spacelike to

$$\mathscr{C}_a + x, \qquad \text{for some } x \in \mathbb{M}^3 . \tag{2.19}$$

(iii) Let us consider two extensions $j^{\mathscr{C}_a^I}$ and $j^{\mathscr{C}_a^{II}}$ of the representation j to two auxiliary algebras $\mathscr{B}^{\mathscr{C}_a^I}$ and $\mathscr{B}^{\mathscr{C}_a^{II}}$ respectively. If A is contained in the algebra $\mathscr{A}(\mathscr{S})^{-w}$ of some simple domain $\mathscr{S}$ spacelike to $\mathscr{C}_a^I + x_I$ and to $\mathscr{C}_a^{II} + x_{II}$, for some $x_I, x_{II} \in \mathbb{M}^3$, then $j^{\mathscr{C}_a^I}(A)$ and $j^{\mathscr{C}_a^{II}}$ coincide:

$$j^{\mathscr{C}_a^I}(A) = j^{\mathscr{C}_a^{II}}(A) . \tag{2.20}$$

For simplicity, we usually suppress the superscript $\mathscr{C}_a$ of the representation $j^{\mathscr{C}_a}$, although it has to be kept in mind that when we write $j(A)$ for operators not contained in the observable algebra $\mathscr{A}$, we are always working with a specific auxiliary cone $\mathscr{C}_a$. For operators such as charge transport operators, belonging to $\mathscr{A}(\mathscr{S})^{-w}$ for some simple domain $\mathscr{S}$, the choice of the auxiliary cone is irrelevant within the limits specified in (iii).

The fact that $\rho_{\mathscr{C}}$ is an endomorphism of $\mathscr{B}^{\mathscr{C}_a}$ (if $\mathscr{C} \subseteq (\mathscr{C}_a + x)'$ for some $x \in \mathbb{M}^3$) allows us to define a *composition of representations*. Let $\rho_{\mathscr{C}_1}^{j_1}$ and $\rho_{\mathscr{C}_2}^{j_2}$ be representations of $\mathscr{A}$ on $\mathscr{H}_1$ unitarily equivalent to j_1 and j_2. If $\mathscr{C}_1 \subseteq (\mathscr{C}_a + x)'$ and $\mathscr{C}_2 \subseteq (\mathscr{C}_a + x)'$, for some x, then, for $A \in \mathscr{A}$, $\rho_{\mathscr{C}_1}^{j_1}(A) \in \mathscr{B}^{\mathscr{C}_a}$ so that

$$\rho_{\mathscr{C}_2}^{j_2} \circ \rho_{\mathscr{C}_1}^{j_1}(A) = \rho_{\mathscr{C}_2}^{j_2}\big(\rho_{\mathscr{C}_1}^{j_1}(A)\big), \qquad A \in \mathscr{A}, \tag{2.21}$$

is well defined. Furthermore, (2.21) is independent of the choice of the auxiliary cone as long as $(\mathscr{C}_a + x) \not\!\wr\, \mathscr{C}_1, \mathscr{C}_2$ for some $x \in \mathbb{M}^3$. We define $j_2 \times j_1$ to be the unitary equivalence class of representations of $\mathscr{A}$ containing the representation $\rho^{j_2}_{\mathscr{C}_2} \circ \rho^{j_1}_{\mathscr{C}_1}(\cdot)$ of $\mathscr{A}$,

$$\rho^{j_2}_{\mathscr{C}_2} \circ \rho^{j_1}_{\mathscr{C}_1} \in j_1 \times j_2 .$$

If $\mathscr{C}_1$ and $\mathscr{C}_2$ are chosen to be spacelike separated, then [20]

$$\rho^{j_2}_{\mathscr{C}_2} \circ \rho^{j_1}_{\mathscr{C}_1}(A) = \rho^{j_1}_{\mathscr{C}_1} \circ \rho^{j_2}_{\mathscr{C}_2}(A) \tag{2.22}$$

so that $j_1 \times j_2$ and $j_2 \times j_1$ define the same unitary equivalence class of representations of $\mathscr{A}$, i.e.,

$$j_1 \times j_2 = j_2 \times j_1 . \tag{2.23}$$

Clearly $j_1 \times j_2$ is localizable in cones. From now on, we use the same letter to denote a representation j of $\mathscr{A}$ on some Hilbert space $\mathscr{H}_j$ and its corresponding unitary equivalence class of representations. The meaning of the symbol will always be clear from the context.

At this point, the analysis of the superselection structure carried out for four-dimensional theories in [17, 18, 19, 20] breaks down in lower dimensional space-times, in particular, in two space-time dimensions, for charges localized in double cones, and in three dimensions if the charges are localized in spacelike cones. This is a reflection of the topological structure of $\mathbb{M}^d$, $d < 4$, and is related to the appearance of *braid statistics* [26], replacing the usual Bose-Einstein and Fermi-Dirac statistics of four-dimensional quantum field theories [17, 18].

These problems have been investigated, within the algebraic framework, by Fredenhagen, Rehren and Schroer, for two-dimensional theories with local charges [22], and in [23, 24] for three-dimensional theories with charges localized in spacelike cones. In this paper, we follow a line of thought similar to the one presented in [24].

We now sketch the proof of several results pertaining to the superselection structure of algebraic field theories which have been derived in [18, 19, 20, 22, 23, 25, 29, 30, 31] under natural assumptions on the physical system under consideration. In subsequent sections, we shall only use those results of the following analysis that are summarized in Definitions 2.1, 2.2 and Property 2.3, below.

First, we note that direct sums and subrepresentations of representations localizable in cones are also localizable in cones. This follows from standard assumptions of quantum field theory, i.e., positivity of the energy, locality and weak additivity, as shown by Borchers in [27].

Next, we make a digression to define the *index of an inclusion of factors*, $N \subseteq M$, as proposed by Jones [28] in the case of type II_1 factors and subsequently generalized in [30, 37] to arbitrary factors. In our case, M will eventually be identified with the algebra

$\mathscr{A}(\mathscr{C})^{-w}$ and N with $\rho(\mathscr{A}(\mathscr{C})^{-w})$, where $\mathscr{C}$ is some spacelike cone chosen to contain the localization region of ρ. We assume (without any essential loss of generality) that $\mathscr{A}(\mathscr{C})^{-w}$ is a factor of type III_1 [47].

Let $N \subseteq M$ be two factors and Ω a joint cyclic and separating vector for N and M – which exists if M and N are properly infinite [38]. (For $M = \mathscr{A}(\mathscr{C})^{-w}$ and $N = \rho(\mathscr{A}(\mathscr{C})^{-w})$, the vacuum vector is such a vector, by the Reeh-Schlieder property.) If J_N^Ω, J_M^Ω are the modular conjugations of N and M with respect to Ω, we define the unitary operator

$$\Gamma_\Omega = J_N^\Omega J_M^\Omega . \tag{2.24}$$

By Tomita's commutation theorem [39],

$$\begin{aligned}\Gamma_\Omega M \Gamma_\Omega^* &= J_N^\Omega J_M^\Omega M J_M^\Omega J_N^\Omega \\ &= J_N^\Omega M' J_N^\Omega \subseteq J_N^\Omega N' J_N^\Omega = N\end{aligned} \tag{2.25}$$

holds and we may therefore define a canonical endomorphism

$$\begin{aligned}\gamma_\Omega &: M \to N \\ A &\to \Gamma_\Omega A \Gamma_\Omega^*, \quad A \in M\end{aligned} \tag{2.26}$$

associated to the triple (M, N, Ω). The properties of the canonical endomorphism (2.26) have been extensively investigated by Longo [32, 33, 34].

If τ is a normal, faithful semifinite trace on M, then $\tau \circ \gamma_\Omega$ is also a trace on M. For a class of traces characterized in [30] and called scalar traces, $\tau \circ \gamma_\Omega$ will be proportional to τ:

$$\tau \circ \gamma_\Omega = \lambda \cdot \tau, \qquad \lambda \in (0; \infty). \tag{2.27}$$

In such a case, we define

$$\mathrm{Ind}_\tau(N, M) = \lambda . \tag{2.28}$$

This index is independent of the particular choice of Ω.

In the general case of two infinite factor $N \subseteq M$, there are no semifinite traces on M. But, given a faithful normal state φ_0 on N and a conditional expectation ε of M onto N, one can set $\varphi = \varphi_0 \circ \varepsilon$ and define the cross-products

$$\tilde{N} = N \times_{\sigma^{\varphi_0}} \mathbb{R} \tag{2.29}$$

$$\tilde{M} = M \times_{\sigma^{\varphi}} \mathbb{R} \tag{2.30}$$

where σ^{φ_0}, σ^{φ} are the modular groups of the states φ_0 and φ, respectively. From this construction we obtain an inclusion of semifinite algebras [40, 30], $\tilde{N} \subseteq \tilde{M}$. The canonical trace $\tilde{\tau}$ on $\tilde{M}$ is then a scalar trace, so that we may define

$$\mathrm{Ind}_{\varepsilon}(N, M) = \mathrm{Ind}_{\tilde{\tau}}(\tilde{N}; \tilde{M}). \tag{2.31}$$

This definition is independent of the choice of φ_0 and coincides with (2.28) in the case of semifinite factors, N, M. The index of N in M, $\mathrm{Ind}(N, M)$, is then given by the minimum of $\mathrm{Ind}_{\varepsilon}(N, M)$ taken over all conditional expectations ε of M onto N (such a minimum exists [30]). For $M = \mathscr{A}(\mathscr{C})^{-w}$ and $N = \rho\big(\mathscr{A}(\mathscr{C})^{-w}\big)$, we define

$$\mathrm{Ind}(\rho) = \mathrm{Ind}\big(\rho\big(\mathscr{A}(\mathscr{C})^{-w}\big), \mathscr{A}(\mathscr{C})^{-w}\big) \tag{2.32}$$

where the right hand side of (2.32) is independent of the particular choice of $\mathscr{C}$ [30].

A theorem of Jones [28] tells us that the value of $\mathrm{Ind}(\rho)$ is contained in the set $\left\{4\cos^2\frac{\pi}{n},\ n = 3, 4, \ldots\right\} \cup [4; \infty]$. In the following, we will restrict our attention to endomorphisms ρ satisfying $\mathrm{Ind}(\rho) < \infty$. Such representations ρ always decompose into a finite direct sum of irreducible representations of finite index [31]. Conversely, the set of representations of finite index is closed under taking direct sums, subrepresentations and products [30, 31].

Longo then proceeds to define the conjugate $\bar{\rho}$ of an irreducible morphism ρ of finite index as the unique *-endomorphism (modulo inner automorphisms) which satisfies

$$\rho \circ \bar{\rho} = \gamma_{\Omega} \tag{2.33}$$

where $\gamma_{\Omega} : \mathscr{A}(\mathscr{C})^{-w} \to \rho\big(\mathscr{A}(\mathscr{C})^{-w}\big)$ is the canonical endomorphism of Eq. (2.26). A beautiful result obtained in [31] characterizes the conjugate morphism $\bar{\rho}$ of an irreducible morphism ρ of finite index as the unique morphism (up to inner automorphisms), such that $\rho\bar{\rho}$ and $\bar{\rho}\rho$ contain the identity as a subrepresentation precisely once. This characterization is analogous to the characterization of the contragradient representation in the dual of a compact group which motivated the original construction of the conjugate charge given by Doplicher, Haag and Roberts [18]. It should be noted here that all the previous results on the structure of superselection sectors had already been obtained by Doplicher, Haag, and Roberts [17, 29] for four-dimensional theories, with the concept of statistical dimension, $d(\rho)$, of a sector ρ replacing the index $\mathrm{Ind}(\rho)$. The relation between the statistical dimension and the index has been fully elucidated by Longo [30] (see also [22]). We will come back to this result later on, since it requires a precise definition of the statistical dimension.

Let us now review the covariance properties of morphisms. We start with a definition generalizing Eq. (2.7).

Definition 2.1. A *-representation j of $\mathscr{A}$ on a separable Hilbert space $\mathscr{H}_j$ is called *covariant* if there exists a strongly continuous, unitary representation of the (covering group of the) Poincaré group implementing the corresponding automorphisms of $\mathscr{A}$:

$$U_j(\Lambda, a) j(A) U_j(\Lambda, a)^* = j(\alpha_{(\Lambda, a)}(A)). \tag{2.34}$$

□

It was shown in [18] that subrepresentations, direct sums as well as products and conjugates of covariant representations are covariant. The proofs, which hold for charges localized in double cones require some slight modifications for theories with charges localized in spacelike cones which are given in [23]. The joint spectrum of the generators of translations is analyzed in [18] (and in [20], leaving aside the assumption of Poincaré covariance of the sectors under consideration; see also [35] for further results). The main outcome is that the set of representations satisfying the relativistic spectrum condition is closed under taking products, direct sums, subrepresentations and conjugates.

Definition 2.2. We denote by L the complete list of unitary equivalence classes of irreducible, covariant, positive-energy representations of $\{\mathscr{A}, \alpha\}$ of finite index localizable in cones.

Property 2.3. (P1) Every covariant, positive-energy representation of $\{\mathscr{A}, \alpha\}$ of finite index is completely reducible into a finite direct sum of irreducible covariant positive-energy representations belonging to L.

(P2) There exists a unique involution ("charge conjugation") on $L: j \in L \to \bar{j} \in L$, such that $j \times \bar{j}$ contains the vacuum representation, 1, of $\mathscr{A}$ precisely once as a subrepresentation. $\bar{j}$ is called the representation (class of representations) conjugate to j.

(P3) The set of covariant, positive-energy representations of $\{\mathscr{A}, \alpha\}$ of finite index which are localizable in cones, is stable under taking direct sums, subrepresentations, conjugates and composition. □

Finally, we remark that the condition $\mathrm{Ind}(\rho) < \infty$ is always satisfied in theories containing only massive particles. This result was derived from the spectral properties of such theories by Fredenhagen [36].

It follows from **(P1)** and **(P3)** that, for i, j in L, $i \times j$ may be decomposed into a direct sum of irreducible representations belonging to L:

$$i \times j = \bigoplus_{k \in L} \bigoplus_{\mu=1}^{N_{ij}^k} k^{(\mu)} \tag{2.35}$$

where $k^{(\mu)}$ is unitarily equivalent to $k \in L$ and $N_{ij}^k \in \{0, 1, 2, \ldots\}$ is the multiplicity of k in $i \times j$. By property **(P2)**, N_{ij}^k can also be interpreted as the multiplicity of 1 in $\bar{k} \times i \times j$. This and (2.23) show that

$$N_{ij}^k = N_{ji}^k = N_{\bar{k}i}^{\bar{j}}. \tag{2.36}$$

We define $|L| \times |L|$ matrices, $\mathbb{N}_j$, $j \in L$ by setting

$$(\mathbb{N}_j)_i^k := N_{ji}^k \in \{0, 1, 2, \dots\} \tag{2.37}$$

($|L|$ is the cardinality of L). One easily sees from (2.23) that

$$\mathbb{N}_1 = \mathbb{1}, \qquad \mathbb{N}_i \mathbb{N}_j = \sum_{k \in L} N_{ij}^k \mathbb{N}_k, \tag{2.38}$$

$$[\mathbb{N}_i; \mathbb{N}_j] = 0 \qquad \forall i, j \in L \tag{2.39}$$

(see [23, 24]). These properties identify the matrices $\{\mathbb{N}_j\}_{j \in L}$ as matrices of *fusion rules.* It is an open problem to classify all possible fusion rules. For examples, see [23, 24, 48], and references therein. Another important property of the fusion matrices $\mathbb{N}_j$ is given by the following lemma.

Lemma 2.4. [22] *$d(\rho_j) := \mathrm{Ind}(\rho_j)^{1/2}$ is the largest eigenvalue of the matrix $\mathbb{N}_j$.* □

The proof is given Sec. 6.

Suppose now that $N_{ji}^k \neq 0$. Then k appears N_{ji}^k times as a subrepresentation of $j \times i$. Equivalently, we may say that the representation $i \circ \rho_{\mathscr{C}}^j$ of $\mathscr{A}$ on $\mathscr{H}_i$ contains N_{ji}^k subrepresentations $k^{(\mu)} \in k$, $\mu = 1, \dots N_{ji}^k$. This and Eq. (2.35) imply that the superselection sector $\mathscr{H}_i$ can be decomposed into a finite direct sum of orthogonal subspaces

$$\mathscr{H}_i = \bigoplus_{k \in L} \bigoplus_{\mu=1}^{N_{ji}^k} \mathscr{H}_i(k; j; \mu) \tag{2.40}$$

with the property that the representation $i \circ \rho_{\mathscr{C}}^j$ of $\mathscr{A}$ on $\mathscr{H}_i(k; j; \mu)$ belongs to class k. Equivalently, Eq. (2.40) implies the existence of N_{ji}^k partial isometries

$$V_\mu^{ik}(\rho_{\mathscr{C}}^j) : \mathscr{H}_k \to \mathscr{H}_i, \qquad \mu = 1, \dots N_{ji}^k \tag{2.41}$$

satisfying the intertwining relation

$$i \circ \rho_{\mathscr{C}}^j(A) V_\mu^{ik}(\rho_{\mathscr{C}}^j) = V_\mu^{ik}(\rho_{\mathscr{C}}^j) k(A) \tag{2.42}$$

and such that

$$V_{\mu_1}^{ik}(\rho_{\mathscr{C}}^j)^* V_{\mu_2}^{ik}(\rho_{\mathscr{C}}^j) = \delta_{\mu_1 \mu_2} \cdot \mathbb{1}|_{\mathscr{H}_k} \tag{2.43}$$

$$V_\mu^{ik}(\rho_{\mathscr{C}}^j) V_\mu^{ik}(\rho_{\mathscr{C}}^j)^* = P_i(k; j; \mu). \tag{2.44}$$

The range of $V_\mu^{ik}(\rho_{\mathscr{C}}^j)$ is the subspace $\mathscr{H}_i(k; j; \mu)$ and $P_i(k; j; \mu)$ is the projection onto this subspace. Finally,

$$\sum_k \sum_{\mu=1}^{N_{ji}^k} V_\mu^{ik}(\rho_{\mathscr{C}}^j) V_\mu^{ik}(\rho_{\mathscr{C}}^j)^* = \mathbb{1}|_{\mathscr{H}_i}. \tag{2.45}$$

If we define a complex vector space $\mathscr{V}_i(\rho^j_{\mathscr{C}})_k$ of operators

$$V : \mathscr{H}_k \to \mathscr{H}_i \tag{2.46}$$

satisfying the intertwining relation

$$i\big(\rho^j_{\mathscr{C}}(A)\big)V = Vk(A) \tag{2.47}$$

then, by (2.40), the range of every $V \in \mathscr{V}_i(\rho^j_{\mathscr{C}})_k$ is contained in the subspace $\bigoplus_{\mu=1}^{N^k_{ji}} \mathscr{H}_i(j;k;\mu)$ of $\mathscr{H}_i$. The vector space $\mathscr{V}_i(\rho^j_{\mathscr{C}})_k$ carries a natural scalar product. For V and W in $\mathscr{V}_i(\rho^j_{\mathscr{C}})_k$ V^*W is an operator from $\mathscr{H}_k$ to $\mathscr{H}_k$ which by (2.47) satisfies

$$k(A)V^*W = V^*Wk(A). \tag{2.48}$$

Since k is irreducible, it follows from Schur's Lemma that

$$V^*W = \lambda\cdot\mathbb{1}, \qquad \lambda \in \mathbb{C}. \tag{2.49}$$

The complex number λ depends anti-linearly on V and linearly on W. Moreover, for $V = W \neq 0$, λ is strictly positive. Hence,

$$\langle V; W\rangle = \lambda \tag{2.50}$$

defines a scalar product on $\mathscr{V}_i(\rho^j_{\mathscr{C}})_k$. With respect to this scalar product, the set $\{V^{ik}_\mu(\rho^j_{\mathscr{C}}),\ \mu = 1, \ldots N^k_{ji}\}$ of partial isometries given in Eqs. (2.41)–(2.44) is an *orthonormal basis* of $\mathscr{V}_i(\rho^j_{\mathscr{C}})_k$. As a complex Hilbert space, $\mathscr{V}_i(\rho^j_{\mathscr{C}})_k$ is isomorphic to $\mathbb{C}^{N^k_{ji}}$ equipped with the usual scalar product.

Finally, we notice that if $V \in \mathscr{V}_i(\rho^j_{\mathscr{C}})_k$ is an intertwiner between the representations $i \circ \rho^j_{\mathscr{C}}$ and k of $\mathscr{A}$, it may be considered to be an intertwiner of the representations $i \circ \rho^j_{\mathscr{C}}$ and k of an auxiliary algebra $\mathscr{B}^{\mathscr{C}_a}$, for some spacelike cone $\mathscr{C}_a$, provided $\mathscr{C}_a + x$ is spacelike to the localization region of $\rho^j_{\mathscr{C}}$, for some $x \in \mathbb{M}^3$. This allows us to apply the intertwining relation (2.47) to operators A in $\mathscr{B}^{\mathscr{C}_a}$, for example to charge transport operators.

3. Bundles of Intertwiners for the Observable Algebra $\mathscr{A}$

3.1. *Irreducible representations*

We wish to interpret the intertwiners $V(\rho) \in \mathscr{V}_i(\rho)_k$ as sections of a *vector bundle over the space of morphisms localized in spacelike cones.* We shall construct such a vector bundle, leaving aside topological questions. In the following, we consider a class, j, of representations of $\mathscr{A}$ as well as the set of irreducible morphisms, $\rho \in j$, localized in spacelike cones.

We start by defining the asymptotic direction of a spacelike cone $\mathscr{C}$. Choose a polar coordinate system (r, θ) in the $\{t = 0\}$-plane and project $\mathscr{C}$ onto the plane to obtain a wedge $\bar{\mathscr{C}}$. Define the asymptotic direction $\theta(\mathscr{C})$ as the θ-coordinate of the line bisecting the wedge $\bar{\mathscr{C}}$. If $\mathscr{C} = (\mathscr{C}_0')'$ is the causal completion of a two-dimensional wedge $\mathscr{C}_0$ then $\theta(\mathscr{C})$ is the θ-coordinate of the line bisecting the wedge $\mathscr{C}_0$. The asymptotic direction $\theta(\mathscr{C})$ is well-defined modulo 2π. If ρ is a morphism localized in the spacelike cone $\mathscr{C}$ then we define the asymptotic direction of ρ as

$$\operatorname{as} \rho = \theta(\mathscr{C}). \tag{3.1}$$

Clearly, two spacelike separated morphisms ρ_1 and ρ_2 satisfy $\operatorname{as}\rho_1 \neq \operatorname{as}\rho_2 \pmod{2\pi}$.

Let us pick a reference cone $\mathscr{C}_r$ of asymptotic direction zero with apex at $x = 0$ so that $\mathscr{C}_r$ is the causal completion of a two-dimensional wedge C_r. Let S be the set of translates and rotates of $\mathscr{C}_r$:

$$S = \{\mathscr{C}_{r(\theta,x)} := R(\theta)\mathscr{C}_r + x, \qquad R(\theta) \in SO(2), x \in \mathbb{M}^3\} \tag{3.2}$$

and $\bar{S}$ the set of spacelike cones contained in some $\mathscr{C}_{r(\theta,x)} \in S$:

$$\bar{S} = \{\mathscr{C}, \mathscr{C} \text{ is a spacelike cone and there exists } \mathscr{C}_{r(\theta,x)} \in S \text{ such that } \mathscr{C} \subseteq \mathscr{C}_{r(\theta,x)}\} \tag{3.3}$$

We restrict our attention to morphisms ρ localized in some spacelike cone $\mathscr{C} \in \bar{S}$:

$$\mathscr{M}_j^{\mathscr{C}_r} := \{\rho \in j \mid \rho \text{ is localized in } \mathscr{C} \in \bar{S}\}. \tag{3.4}$$

Two subsets $\mathscr{M}_j^I$ and $\mathscr{M}_j^{II}$ of $\mathscr{M}_j^{\mathscr{C}_r}$ are defined as follows. Consider the spacelike complement of $\mathscr{C}_r \cup \mathscr{C}_{r(\pi,0)}$, $\mathscr{C}_{r(\pi,0)}$ being the π-rotate of $\mathscr{C}_r$. It consists of two distinct, connected regions whose projection on the $\{t = 0\}$-plane is sketched in Fig. 3.1. We choose two auxiliary cones $\mathscr{C}_a^I$ and $\mathscr{C}_a^{II}$ spacelike to $\mathscr{C}_r$ and $\mathscr{C}_{r(\pi,0)}$, each auxiliary cone lying in one of the connected components of $(\mathscr{C}_r \cup \mathscr{C}_{r(\pi,0)})'$ (see Fig. 3.1).

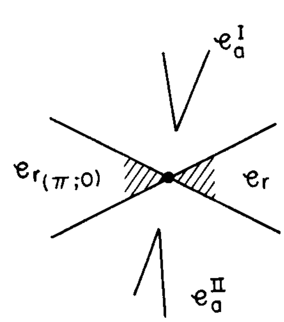


Fig. 3.1

Define

$$S^{\#} := \{\mathscr{C}_{r(\theta,x)} \in S \text{ such that } \mathscr{C}_{r(\theta,x)} \text{ is spacelike to } \mathscr{C}_a^{\#} + y \text{ for some } y \in \mathbb{M}^3\} \tag{3.5}$$

$$\bar{S}^{\#} := \{\mathscr{C} \in \bar{S} \text{ such that } \mathscr{C} \subseteq \mathscr{C}_{r(\theta,x)} \text{ for some } \mathscr{C}_{r(\theta,x)} \in S^{\#}\} \tag{3.6}$$

for $\# = I, II$ and

$$\mathscr{M}_j^{\#} := \{\rho \mid \rho \in \mathscr{M}_j^{\mathscr{C}_r}, \rho \text{ is localized in } \mathscr{C} \in \bar{S}^{\#}\}.$$

Clearly, the two sets $\mathscr{M}_j^{\#}$, $\# = I, II$ cover $\mathscr{M}_j^{\mathscr{C}_r}$. The intersection $\mathscr{M}_j^I \cap \mathscr{M}_j^{II}$ is described as follows. Denote by $\mathscr{P}(\mathscr{C}_r)$ the set of spacelike cones contained in $S^I \cap S^{II}$ which can be joined to the reference cone $\mathscr{C}_r$ by a continuous path inside $S^I \cap S^{II}$. Let

$$\bar{\mathscr{P}}(\mathscr{C}_r) := \{\mathscr{C} \in \bar{S}^I \cap \bar{S}^{II}, \text{ such that } \mathscr{C} \subseteq \mathscr{C}_{r(\theta,x)} \text{ for some } \mathscr{C}_{r(\theta,x)} \in \mathscr{P}(\mathscr{C}_r)\} \tag{3.7}$$

and

$$\mathscr{M}(\bar{\mathscr{P}}(\mathscr{C}_r)) := \{\rho \in \mathscr{M}_j^{\mathscr{C}_r} \text{ such that } \rho \text{ is localized in } \mathscr{C} \in \bar{\mathscr{P}}(\mathscr{C}_r)\}. \tag{3.8}$$

Then

$$\mathscr{M}_j^I \cap \mathscr{M}_j^{II} = \mathscr{M}(\bar{\mathscr{P}}(\mathscr{C}_r)) \coprod \mathscr{M}(\bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)})) \tag{3.9}$$

where the symbol $\coprod$ denotes the disjoint union. In the following, we will always work with morphisms localized in a common "coordinate chart" $\mathscr{M}_j^I$ or $\mathscr{M}_j^{II}$ of $\mathscr{M}_j^{\mathscr{C}_r}$. In each one of the sets $\mathscr{M}_j^{\#}$, $\# = I, II$, it is possible to define the asymptotic direction $\mathrm{as}_{\#}(\rho)$ of a morphism ρ unambiguously. This is most easily seen in the following figure:

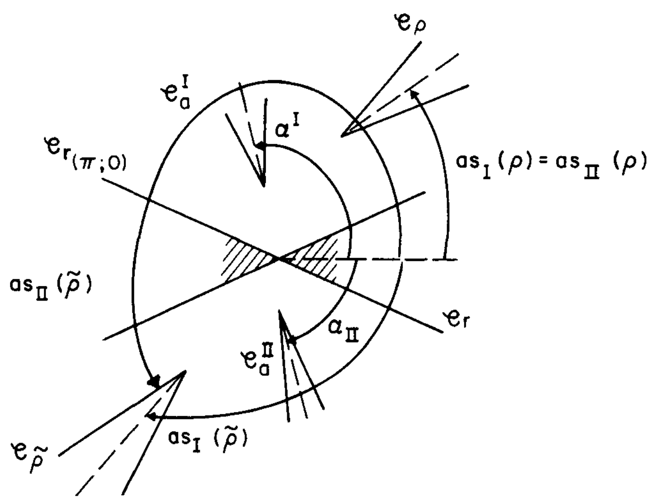


Fig. 3.2

where ρ and $\tilde{\rho}$ are morphisms localized in $\mathscr{C}_\rho$ and $\mathscr{C}_{\tilde{\rho}}$ respectively. The geometrical situation depicted in Fig. 3.2 corresponds to imposing

$$\begin{aligned}
&\text{(i)} && \mathrm{as}_I(\rho) \in (-2\pi + \alpha^I; \alpha^I) \\
& && \text{for } \rho \in \mathscr{M}_j^I, \qquad \text{where } \alpha^I = \theta(\mathscr{C}_a^I) \in (0;\pi). \\
&\text{(ii)} && \mathrm{as}_{II}(\rho) \in (\alpha^{II}; 2\pi + \alpha^{II}) \\
& && \text{for } \rho \in \mathscr{M}_j^{II}, \qquad \text{where } \alpha^{II} = \theta(\mathscr{C}_a^{II}) \in (-\pi;0).
\end{aligned} \tag{3.10}$$

The corresponding angle functions $\theta_\#(\mathscr{C})$, $\# = I, II$ for spacelike cones $\mathscr{C} \in \bar{S}$ are defined similarly. For a morphism $\rho \in \mathscr{M}_j^I \cap \mathscr{M}_j^{II}$ we have

$$\mathrm{as}_{II}(\rho) - \mathrm{as}_I(\rho) = \begin{cases} 2\pi & \text{if } \rho \in \mathscr{M}(\bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)})) \\ 0 & \text{if } \rho \in \mathscr{M}(\bar{\mathscr{P}}(\mathscr{C}_r)) \end{cases} \tag{3.11}$$

(see Fig. 3.2).

If two morphisms ρ_1 and ρ_2 are contained in the same subset $\mathscr{M}_j^\#$, $\# = I$ or II, then there exists an intertwiner $\Gamma^\#_{\rho_1\rho_2}$ such that

$$\rho_1(A)\Gamma^\#_{\rho_1\rho_2} = \Gamma^\#_{\rho_1\rho_2}\rho_2(A) \tag{3.12}$$

holds and $\Gamma^\#_{\rho_1\rho_2}$ belongs to the auxiliary algebra $\mathscr{B}^\#$,

$$\mathscr{B}^\# := \mathscr{B}^{\mathscr{C}_a^\#}. \tag{3.13}$$

Moreover, any other unitary operator Γ intertwining ρ_1 and ρ_2 differs from $\Gamma^\#_{\rho_1\rho_2}$ only by a phase factor: since $\Gamma^*\Gamma^\#_{\rho_1\rho_2}$ intertwines ρ_2 with itself and the representation ρ_2 is irreducible on the vacuum sector, it follows that

$$\Gamma^*\Gamma^\#_{\rho_1\rho_2} = e^{i\theta}\cdot \mathbb{1}, \tag{3.14}$$

for some $\theta \in [0, 2\pi)$.

Let the unitary and projective unitary groups of the algebra $\mathscr{B}^\#$, $\# = I, II$ be defined by

$$\mathscr{U}^\# := \{\Gamma \in \mathscr{B}^\# \mid \Gamma\Gamma^* = \mathbb{1}\} \tag{3.15}$$

and

$$P\mathscr{U}^\# := \mathscr{U}^\#/U(1). \tag{3.16}$$

In the definition of $P\mathscr{U}^\#$, we identify unitary operators in $\mathscr{U}^\#$ differing only by a phase factor. If we choose a reference morphism $\rho_r \in \mathscr{M}_j^\#$, then the set of morphisms $\rho \in \mathscr{M}_j^\#$ is in one-to-one correspondence with a subset of $P\mathscr{U}^\#$:

$$\mathscr{M}_j^\# \ni \rho \leftrightarrow [\Gamma^\#_{\rho\rho_r}] \in P\mathscr{U}^\# \tag{3.17}$$

where $\Gamma^{\#}_{\rho\rho_r} \in \mathscr{U}^{\#}$ satisfies

$$\rho(A)\Gamma^{\#}_{\rho\rho_r} = \Gamma^{\#}_{\rho\rho_r}\rho_r(A) \tag{3.18}$$

and the square bracket [ ] denotes the equivalence class of $\Gamma^{\#}_{\rho\rho_r}$ in $P\mathscr{U}^{\#}$. It will be convenient to use this fact later on.

Let G be the (covering group of the) rotation and translation group in three-dimensional space. Then G is a subgroup of the (covering group of the) Poincaré group $\tilde{P}^{\uparrow}_{+}$. We label elements of G by $g = (\theta, x)$, where (θ, x) corresponds to the action

$$g \cdot y = (\theta, x)y = R(\theta)y + x, \qquad y \in \mathbb{M}^3 \tag{3.19}$$

of G on $\mathbb{M}^3$.

For a covariant representation j of the observable algebra $\mathscr{A}$ (see Definition 2.1), we can define an action of G on $\mathscr{M}_j^{\mathscr{C}_r}$ as follows. If $\rho \in j$ and $(\Lambda, a) \in \tilde{\mathscr{P}}^{\uparrow}_{+}$, set

$$U_\rho(\Lambda, a) := V_{\mathscr{C}} U_j(\Lambda, a) V_{\mathscr{C}}^{*}, \tag{3.20}$$

where $V_{\mathscr{C}}$ is the isometry, unique up to a phase, used to define ρ (see Eq. (2.12)). Following Doplicher, Haag and Roberts [18] one constructs the cocycles

$$\Gamma(\rho; g) = U_1(g)U_\rho(g^{-1}), \qquad g \in \tilde{P}^{\uparrow}_{+} \tag{3.21}$$

and the Poincaré transformations of ρ:

$$\rho_g(A) := \Gamma(\rho; g)\rho(A)\Gamma(\rho; g)^{*}. \tag{3.22}$$

One easily checks that if ρ is localized in the cone $\mathscr{C}$, then ρ_g is localized in

$$\mathscr{C}_g = \{g \cdot x | x \in \mathscr{C}\} \tag{3.23}$$

and hence, for $\rho \in \mathscr{M}_j^{\mathscr{C}_r}$ and $g \in G$, $\rho_g \in \mathscr{M}_j^{\mathscr{C}_r}$. The map

$$\begin{aligned} \eta : G \times \mathscr{M}_j^{\mathscr{C}_r} &\to \mathscr{M}_j^{\mathscr{C}_r} \\ (g; \rho) &\to \rho_g \end{aligned} \tag{3.24}$$

defines an action of G on $\mathscr{M}_j^{\mathscr{C}_r}$, as one easily checks.

In particular the "cocycle identity"

$$\eta(g_1 g_2; \rho) = \eta(g_1; \rho_{g_2}) \tag{3.25}$$

holds. In the following, we only use the translation and rotation covariance of the representations in L. For $j \in L$, let $U_j(2\pi)$ be the unitary operator representing a rotation by 2π. Clearly, $U_j(2\pi)$ commutes with $j(\mathscr{A})$ and, since j is irreducible it is a multiple of the identity:

$$U_j(2\pi) = e^{2\pi i s_j} \cdot \mathbb{1}\,, \tag{3.26}$$

where s_j is called the *spin* of the representation j. It labels irreducible representations of the covering group of the rotations $\widetilde{SO}(2) = \mathbb{R}$ and hence can be any real number in the interval $[0; 1)$. If the representation j contains one-particle states, then the spins of the corresponding particles will be equal to s_j (mod $\mathbb{Z}$).

We introduce the following notation:

$$J_{ijk} := \{V(\rho) \in \mathscr{V}_i(\rho)_k : \rho \in \mathscr{M}_j^{\mathscr{C}_r}\}\,. \tag{3.27}$$

The space J_{ijk}, equipped with the projection

$$\begin{aligned} \mathrm{pr} : J_{ijk} &\to \mathscr{M}_j^{\mathscr{C}_r} \\ V(\rho) &\to \rho \end{aligned} \tag{3.28}$$

has fibres $\mathrm{pr}^{-1}(\rho) = \mathscr{V}_i(\rho)_k$ isomorphic to $\mathbb{C}^{N_{ji}^k}$. We can give the two sets $\mathscr{N}_j^{\#} = \mathrm{pr}^{-1}(\mathscr{M}_j^{\#})$, $\# = I, II$ a local product structure,

$$\mathscr{N}_j^{\#} \cong \mathscr{M}_j^{\#} \times \mathbb{C}^{N_{ji}^k}, \tag{3.29}$$

as we now explain. Choose a reference morphism ρ_r localized in the reference cone $\mathscr{C}_r$, $\rho_r \in \mathscr{M}_j^I \cap \mathscr{M}_j^{II}$ and an orthonormal basis, $\{V_\alpha^{ik}(\rho_r)\}_{\alpha=1}^{N_{ji}^k}$, of the fibre $\mathscr{V}_i(\rho_r)_k$ over the reference morphism ρ_r. Let $\rho \in \mathscr{M}_j^{\#}$, $\# = I, II$ be localized in a cone $\mathscr{C}$ of asymptotic direction $\mathrm{as}_{\#}\rho$. If $[\Gamma_{\rho\rho_r}^{\#}]$ is the element of $P\mathscr{U}^{\#}$ corresponding to ρ, we pick a representative $\Gamma_{\rho\rho_r}^{\#} \in \mathscr{U}^{\#}$ of $[\Gamma_{\rho\rho_r}^{\#}]$. Then the operators

$$V_{\alpha\#}^{ik}(\rho) := i(\Gamma_{\rho\rho_r}^{\#}) V_\alpha^{ik}(\rho_r), \qquad \alpha = 1, \ldots, N_{ji}^k, \tag{3.30}$$

satisfy the intertwining relations (2.47) and are thus elements of the fibre $\mathscr{V}_i(\rho)_k$; the representation i occurring in Eq. (3.30) is the extension of the representation i of $\mathscr{A}$ to $\mathscr{B}^{\#}$, $\# = I, II$ (see Sec. 2, (2.19)), and $V_{\alpha\#}^{ik}(\rho)$ intertwines the representations $i \circ \rho$ and k of $\mathscr{B}^{\#}$, $\# = I, II$ (see the last paragraph of Sec. 2). Furthermore, the intertwiners $V_{\alpha\#}^{ik}(\rho)$, $\alpha = 1, \ldots, N_{ji}^k$ form an orthonormal basis in $\mathscr{V}_i(\rho)_k$ under the scalar product (2.50). Hence, the mapping

$$\begin{aligned} \mathscr{N}_j^{\#} &\to \mathscr{M}_j^{\#} \times \mathbb{C}^{N_{ji}^k} \\ V(\rho) &\to \left(\rho; \{\langle V_{\alpha\#}^{ik}(\rho); V(\rho)\rangle\}_{\alpha=1}^{N_{ji}^k}\right) \end{aligned} \tag{3.31}$$

defines coordinates of $V(\rho)$ in $\mathscr{M}_j^{\#} \times \mathbb{C}^{N_{ji}^k}$. There is still a phase ambiguity in Eq. (3.31), due to the fact that we have not yet specified which representative $\Gamma_{\rho\rho_r}^{\#}$ of the equivalence class $[\Gamma_{\rho\rho_r}^{\#}]$ is chosen. The choice of $\Gamma_{\rho\rho_r}^{\#}$ proceeds as follows. Since ρ is localized in $\mathscr{C}$, there exists some cone $\mathscr{C}_{r(\theta_\#, x)} \in S$ such that $\mathscr{C} \subseteq \mathscr{C}_{r(\theta_\#, x)}$ holds, by definition of

$\mathscr{M}_j^{\mathscr{C}_r}$. Choose such a cone $\mathscr{C}_{r(\theta_\#, x)}$ where $\theta_\#$ is fixed by the convention (3.10). Rotate and translate the reference morphism ρ_r to the cone $\mathscr{C}_{r(\theta_\#, x)}$ by using the Poincaré cocycles corresponding to a successive rotation $\theta_\#$ and translation x of ρ_r:

$$\rho_{r(\theta_\#, x)}(A) = \Gamma(\rho_r; (\theta_\#, x))\rho_r(A)\Gamma(\rho_r; (\theta_\#, x))^* . \tag{3.32}$$

The morphism so obtained is localized in $\mathscr{C}_{r(\theta_\#, x)}$. Since the localization cone $\mathscr{C}$ of ρ is contained in $\mathscr{C}_{r(\theta_\#, x)}$, there exists an intertwiner $V_\# \in \mathscr{A}(\mathscr{C}_{r(\theta_\#, x)})^{-w}$ between ρ and $\rho_{r(\theta_\#, x)}$:

$$\rho(A)V_\# = V_\# \rho_{r(\theta_\#, x)}(A). \tag{3.33}$$

We then choose the intertwiner $\Gamma^\#_{\rho\rho_r}$ between ρ and ρ_r as

$$\Gamma^\#_{\rho\rho_r} := V_\# \Gamma(\rho_r; (\theta_\#, x)), \qquad \# = I, II. \tag{3.34}$$

If $\rho \in \mathscr{M}_j^I \cap \mathscr{M}_j^{II}$ we require further that $\mathscr{C}_{r(\theta_I, x)}$ and $\mathscr{C}_{r(\theta_{II}, y)}$ be chosen to coincide

$$\mathscr{C}_{r(\theta_I, x)} = \mathscr{C}_{r(\theta_{II}, y)} =: \mathscr{C}_{r(\theta, x)} \tag{3.35}$$

and that

$$V_I = V_{II} \tag{3.36}$$

holds. The remaining phase ambiguity, due to the fact that there are, in general, many cones $\mathscr{C}_{r(\theta, x)}$ such that $\mathscr{C} \subseteq \mathscr{C}_{r(\theta, x)}$ and that $V_\#$ is unique only up to a phase, $\# = I, II$, is irrelevant in the following.

To complete the description of the bundle J_{ijk}, we have to specify its transition functions. This also determines the group of the bundle J_{ijk}. We evaluate the transition functions on morphisms $\rho \in \mathscr{M}_j^I \cap \mathscr{M}_j^{II}$ localized inside spacelike cones $\mathscr{C}_{r(\theta, x)}$. As we will see, they depend only on whether $\mathscr{C}_{r(\theta, x)}$ belongs to $\mathscr{P}(\mathscr{C}_r)$ or to $\mathscr{P}(\mathscr{C}_{r(\pi, 0)})$. These two different situations are depicted in Fig. 3.3 (a) and (b) below.

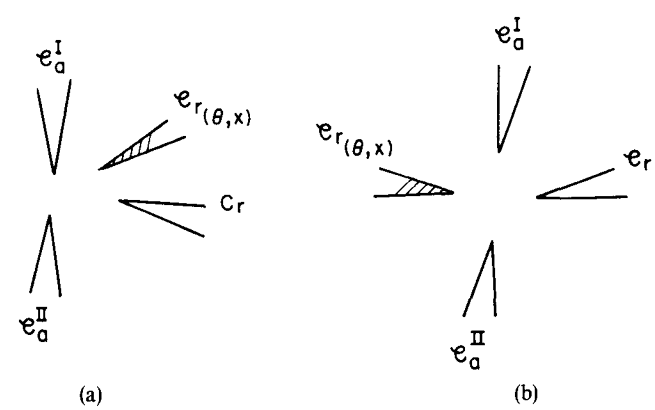


Fig. 3.3

Let $V(\rho) \in J_{ijk}$ be an intertwiner in the fibre over ρ. Its coordinates are given by

$$\left(\rho; \{\langle i(V)i(\Gamma(\rho_r;(\theta_I, x)))V_\alpha^{ik}(\rho_r); V(\rho)\rangle\}_{\alpha=1}^{N_{ji}^k}\right) \tag{3.37}$$

in $\mathcal{N}_j^I$ and by

$$\left(\rho; \{\langle i(V)i(\Gamma(\rho_r;(\theta_{II}, x)))V_\alpha^{ik}(\rho_r); V(\rho)\rangle\}_{\alpha=1}^{N_{ji}^k}\right) \tag{3.38}$$

in $\mathcal{N}_j^{II}$, where we used the fact that $V_I = V_{II}$ to suppress the Latin index I, II (see Eq. (3.36)). We compute the transformation mapping the vector $a_I = \{\langle i(V)i(\Gamma(\rho_r;(\theta_I, x))) \times V_\alpha^{ik}(\rho_r); V(\rho)\rangle\}_{\alpha=1}^{N_{ji}^k}$ to $a_{II} = \{\langle i(V)i(\Gamma(\rho_r;(\theta_{II}, x)))V_\alpha^{ik}(\rho_r); V(\rho)\rangle\}_{\alpha=1}^{N_{ji}^k}$: a_I and a_{II} are the coordinate vectors of $V(\rho)$ with respect to the orthonormal bases $V_{\alpha I}^{ik}(\rho) = i(V)i(\Gamma(\rho_r;(\theta_{I,x})))V_\alpha^{ik}(\rho_r)$ and $V_{\alpha II}^{ik}(\rho) = i(V)i(\Gamma(\rho_r;(\theta_{II,x})))V_\alpha^{ik}(\rho_r)$ of $\mathscr{V}_i(\rho)_k$. Hence, the transition matrix $b_{ik}(II, I) = (b_{ik}(II, I)_\alpha^\beta)$ is given by

$$V_{\alpha II}^{ik}(\rho) = \sum_{\beta=1}^{N_{ji}^k} b_{ik}(II, I)_\alpha^\beta V_{\beta I}^{ik}(\rho) \tag{3.39}$$

where $b_{ik}(II, I)_\alpha^\beta = \langle V_{\beta I}^{ik}(\rho); V_{\alpha II}^{ik}(\rho)\rangle$. Clearly, $b_{ik}(II, I) \in \mathscr{U}_{\mathbb{C}}(N_{ji}^k)$, since it relates two orthonormal bases in $\mathbb{C}^{N_{ji}^k}$. The formula for $b_{ik}(II, I)$ is given in the next theorem.

Theorem 3.1. *The transition matrix $b_{ik}(II, I)$ introduced in Eq. (3.39) for the morphism $\rho \in \mathcal{M}_j^I \cap \mathcal{M}_j^{II}$ is given by*

$$b_{ik}(II, I) = \begin{cases} e^{2\pi i(s_k - s_i)} \cdot \mathbb{1} & \text{if } \rho \in \mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)})) \\ \mathbb{1} & \text{if } \rho \in \mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_r)), \end{cases} \tag{3.40}$$

that is, $b_{ik}(II, I)$ depends only on the two sectors i, k and on the two disjoint components $\mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_r))$, $\mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)}))$ of $\mathcal{M}_j^I \cap \mathcal{M}_j^{II}$, but not on ρ. □

Proof. We evaluate the scalar product

$$\begin{aligned} b_{ik}(II, I)_\alpha^\beta &= \langle V_{\beta I}^{ik}(\rho); V_{\alpha II}^{ik}(\rho)\rangle = V_{\beta I}^{ik}(\rho)^* V_{\alpha II}^{ik}(\rho) \\ &= V_\beta^{ik}(\rho_r)^* i(\Gamma(\rho_r;(\theta_{II,x})))^* i(V)^* i(V) i(\Gamma(\rho_r;(\theta_{I,x})))V_\alpha^{ik}(\rho_r). \end{aligned} \tag{3.41}$$

The unitary operator

$$i(\Gamma(\rho_r;(\theta_{II,x})))^* i(V)^* \tag{3.42}$$

on the Hilbert space $\mathscr{H}_i$ is given by extending the representation i of $\mathscr{A}$ to the algebra $\mathscr{B}^{II}$ (see the paragraph following Eq. (3.30)). Similarly,

$$i(V)i(\Gamma(\rho_r;(\theta_{I,x}))) \tag{3.43}$$

is given by the extension of i to $\mathscr{B}^I$. These two extended representations need not coincide in general. Nevertheless, since V belongs to the algebra $\mathscr{A}(\mathscr{C}_{r(\theta,x)})^{-w}$ where

$\mathscr{C}_{r(\theta,x)}$ is spacelike to $\mathscr{C}_\alpha^I + y$ and to $\mathscr{C}_\alpha^{II} + z$, for some $y, z \in \mathbb{M}^3$, we can apply Eq. (2.20) to conclude that

$$i(V)^* i(V) = \mathbb{1} \tag{3.44}$$

in Eq. (3.41). We can now rewrite

$$b_{ik}(II, I)_\alpha^\beta = V_\beta^{ik}(\rho_r)^* i(\Gamma(\rho_r; (\theta_{II,x}))^*) i(\Gamma(\rho_r; (\theta_{I,x}))) V_\alpha^{ik}(\rho_r). \tag{3.45}$$

The simple argument which allowed us to compute $i(V)^* i(V)$ will in general fail for the Poincaré cocycles of Eq. (3.45), because of their weaker localization properties. Nevertheless we can calculate $b_{ik}(II, I)_\alpha^\beta$ as follows. It has been shown in [18, 23] that if two superselection sectors i and j are covariant, then their product is also covariant, and a simple calculation [18, 23] shows that, for $g = (\theta_I, x)$ or (θ_{II}, x),

$$U_{i\circ\rho}(g^{-1}) = U_i(g)^* \cdot i(\Gamma(\rho; g)) \tag{3.46}$$

holds; (see Eq. (3.21) for the definition of the Poincaré cocycles $\Gamma(\rho; g)$). Hence we may write

$$i(\Gamma(\rho_r; (\theta_I, x))) V_\alpha^{ik}(\rho_r) = U_i(\theta_I, x) U_{i\circ\rho_r}((\theta_I, x)^{-1}) V_\alpha^{ik}(\rho_r) \tag{3.47}$$

$$i(\Gamma(\rho_r; (\theta_{II}, x))) V_\beta^{ik}(\rho_r) = U_i(\theta_{II}, x) U_{i\circ\rho_r}((\theta_{II}, x)^{-1}) V_\beta^{ik}(\rho_r) \tag{3.48}$$

where $(\theta_\#, x)^{-1}$ denotes the inverse of rotation by $\theta_\#$ followed by the translation by $-x$ in G. We can now apply a basic result of Doplicher, Haag and Roberts [18] stating that an intertwiner between two covariant representations of $\mathscr{A}$ automatically intertwines the corresponding unitary representations of the Poincaré group:

$$U_{i\circ\rho_r}(g) V_\alpha^{ik}(\rho_r) = V_\alpha^{ik}(\rho_r) U_k(g), \; g \in \tilde{P}_+^\uparrow. \tag{3.49}$$

Thus we may rewrite Eqs. (3.47), (3.48) as

$$i(\Gamma(\rho_r; (\theta_I, x))) V_\alpha^{ik}(\rho_r) = U_i(\theta_I, x) V_\alpha^{ik}(\rho_r) U_k((\theta_I, x)^{-1}) \tag{3.50}$$

$$i(\Gamma(\rho_r; (\theta_{II}, x))) V_\beta^{ik}(\rho_r) = U_i(\theta_{II}, x) V_\beta^{ik}(\rho_r) U_k((\theta_{II}, x)^{-1}). \tag{3.51}$$

Inserting these equations in (3.45), we find,

$$\begin{aligned} b_{ik}(II, I)_\alpha^\beta = {} & U_k((\theta_{II}, x)^{-1})^* V_\beta^{ik}(\rho_r)^* U_i(\theta_{II}, x)^* U_i(\theta_I, x) \\ & \times V_\alpha^{ik}(\rho_r) U_k((\theta_I, x)^{-1}) \end{aligned} \tag{3.52}$$

or

$$b_{ik}(II, I)_\alpha^\beta = U_k(\theta_{II}, x) V_\beta^{ik}(\rho_r)^* U_i((\theta_{II}, x)^{-1}) U_i(\theta_I, x)$$
$$\times V_\alpha^{ik}(\rho_r) U_k((\theta_I, x)^{-1}). \tag{3.53}$$

An easy calculation shows that

$$U_i((\theta_{II}, x)^{-1}) U_i(\theta_I, x) = U_i(0, -R(-\theta_{II})x) U_i(\theta_I - \theta_{II}, 0)$$
$$\times U_i(0; R(-\theta_I)x). \tag{3.54}$$

Using Eq. (3.11), we then see that

$$U_i((\theta_{II}, x)^{-1}) U_i(\theta_I, x) = \begin{cases} e^{-2\pi i s_i} & \text{if } \rho \in \mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)})) \\ 1 & \text{if } \rho \in \mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_r)). \end{cases} \tag{3.55}$$

By the orthonormality of $\{V_\alpha^{ik}(\rho_r)\}_{\alpha=1}^{N_{ij}^k}$,

$$V_\beta^{ik}(\rho_r)^* V_\alpha^{ik}(\rho_r) = \delta_{\alpha\beta}. \tag{3.56}$$

We also find

$$U_k(\theta_{II}, x) U_k((\theta_I, x)^{-1}) = \begin{cases} e^{2\pi i s_k} & \text{if } \rho \in \mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)})) \\ 1 & \text{if } \rho \in \mathcal{M}(\bar{\mathscr{P}}(\mathscr{C}_r)). \end{cases} \tag{3.57}$$

Combining (3.53)–(3.57) we obtain (3.40). ■

This completes our description of the vector bundle structure of J_{ijk} : J_{ijk} is a nontrivial $U(1)$-bundle, a fact which reflects the topology of the manifold of spacelike asymptotic directions of space-time.

Finally, we remark that all considerations in Sec. 3.1 are independent of the particular choice of the reference cone $\mathscr{C}_r$ and of the auxiliary cones $\mathscr{C}_a^I$ and $\mathscr{C}_a^{II}$. This will also hold in the following sections.

3.2. *The intertwiner bundle of a product representation*

We construct an intertwiner bundle $J_{lp\times qk}$ for a product representation $p \times q$, where $p, q \in L$, in analogy with our construction in Sec. 3.1. Considering the vector space of intertwiners $\mathscr{V}_l(\rho^p \circ \rho^q)_k$ for the product $\rho^p \circ \rho^q$ of irreducible morphisms $\rho^p \in \mathcal{M}_p^{\mathscr{C}_r}$, $\rho^q \in \mathcal{M}_q^{\mathscr{C}_r}$, we observe that the full reducibility of composed representations (property (P1) of Sec. 2) implies that it can be constructed in terms of the intertwiner spaces $\mathscr{V}_l(\rho^p)_m$ and $\mathscr{V}_m(\rho^q)_k$ of irreducible morphisms ρ^p, ρ^q. This will allow us to calculate the transition functions of $J_{lp\times qk}$ in terms of the transition functions of J_{lpm}, J_{mqk}.

Consider the representation $l \circ \rho^p \circ \rho^q$ of the observable algebra $\mathscr{A}$. This representation can be decomposed into a direct sum

$$l \circ \rho^p \circ \rho^q \cong \bigoplus_{k\in L} \bigoplus_{\mu=1}^{N_{lp\times q}^k} k^{(\mu)} \tag{3.58}$$

where $N^k_{lp\times q}$ denotes the multiplicity of the representation k in $l\circ\rho^p\circ\rho^q$. Hence, we may define a complex vector space $\mathscr{V}_l(\rho^p\circ\rho^q)_k$ of intertwiners

$$V(\rho^p\circ\rho^q):\mathscr{H}_k\to\mathscr{H}_l \tag{3.59}$$

satisfying

$$l\big(\rho^p\circ\rho^q(A)\big)V(\rho^p\circ\rho^q)=V(\rho^p\circ\rho^q)k(A). \tag{3.60}$$

This complex vector space is isomorphic to $\mathbb{C}^{N^k_{lp\times q}}$, for arbitrary $\rho^p\in\mathscr{M}^{\mathscr{C}_r}_p$, $\rho^q\in\mathscr{M}^{\mathscr{C}_r}_q$.

Performing the successive decompositions

$$l\circ\rho^p\cong\bigoplus_m\bigoplus_{\alpha=1}^{N^m_{lp}}m^{(\alpha)}, \tag{3.61}$$

$$m\circ\rho^q\cong\bigoplus_k\bigoplus_{\beta=1}^{N^k_{mq}}k^{(\beta)}, \tag{3.62}$$

and comparing these results with (3.58) we conclude that

$$N^k_{lp\times q}=\sum_{m\in L}N^m_{lp}N^k_{mq} \tag{3.63}$$

where the sum on the right hand side extends over the finite subset of representations in L satisfying $N^m_{lp}\cdot N^k_{mq}\neq 0$. Let $\{V^{lm}_\alpha(\rho^p)\}^{N^m_{lp}}_{\alpha=1}$, $\{V^{mk}_\beta(\rho^q)\}^{N^k_{mq}}_{\beta=1}$ be orthonormal bases of $\mathscr{V}_l(\rho^p)_m$ and $\mathscr{V}_m(\rho^q)_k$ respectively. Then

$$\{V^{lm}_\alpha(\rho^p)V^{mk}_\beta(\rho^q),\ \alpha=1,\dots N^m_{lp},\ \beta=1,\dots N^k_{mq},\ m\in L\}, \tag{3.64}$$

is an orthonormal system in $\mathscr{V}_l(\rho^p\circ\rho^q)_k$, since each intertwiner satisfies Eq. (3.60). Here $\{V^{lm}_\alpha(\rho^p)\}^{N^m_{lp}}_{\alpha=1}$ and $\{V^{mk}_\beta(\rho^q)\}^{N^k_{mq}}_{\beta=1}$ are understood to intertwine the corresponding representations of an auxiliary algebra $\mathscr{B}^{\mathscr{C}_a}$, where $\mathscr{C}_a$ is spacelike to the localization cones of ρ^p, ρ^q. Equation (3.63) implies that (3.64) is really an orthonormal basis of $\mathscr{V}_l(\rho^p\circ\rho^q)_k$, or, equivalently

$$\mathscr{V}_l(\rho^p\circ\rho^q)_k=\bigoplus_m\mathscr{V}_l(\rho^p)_m\otimes\mathscr{V}_m(\rho^q)_k \tag{3.65}$$

$$\mathbb{C}^{N^k_{lp\times q}}=\bigoplus_m\mathbb{C}^{N^m_{lp}}\otimes\mathbb{C}^{N^k_{mq}}. \tag{3.66}$$

Naturally, Eqs. (3.65), (3.66) generalize to arbitrary order products $\rho^p\circ\rho^q\circ\cdots\circ\rho^s$.

We now pick a reference cone $\mathscr{C}_r$ of asymptotic direction zero, two reference morphisms ρ^p_r, ρ^q_r localized in $\mathscr{C}_r$ and two auxiliary cones $\mathscr{C}^I_a$ and $\mathscr{C}^{II}_a$, as in Sec. 3.1. We use the following notation:

$$\mathscr{M}^{\#}_{p\times q}:=\{\rho^p\circ\rho^q,\rho^p\in\mathscr{M}^{\#}_p,\rho^q\in\mathscr{M}^{\#}_q\},\qquad \#=I,II \tag{3.67}$$

$$\mathcal{M}^{\mathscr{C}_r}_{p\times q} := \mathcal{M}^{I}_{p\times q} \cup \mathcal{M}^{II}_{p\times q} \tag{3.68}$$

so that $\mathcal{M}^{I}_{p\times q} \cap \mathcal{M}^{II}_{p\times q}$ is given by

$$\mathcal{M}^{I}_{p\times q} \cap \mathcal{M}^{II}_{p\times q} = \{\rho^p \circ \rho^q | \rho^p \in \mathcal{M}^{I}_{p} \cap \mathcal{M}^{II}_{p}, \quad \rho^q \in \mathcal{M}^{I}_{q} \cap \mathcal{M}^{II}_{q}\}. \tag{3.69}$$

Next, we define the space

$$J_{lp\times qk} = \{V(\rho^p \circ \rho^q) \in \mathscr{V}_l(\rho^p \circ \rho^q)_k,\ \rho^p \circ \rho^q \in \mathcal{M}^{\mathscr{C}_r}_{p\times q}\} \tag{3.70}$$

and the projection

$$\begin{aligned} \mathrm{pr} : J_{lp\times qk} &\to \mathcal{M}^{\mathscr{C}_r}_{p\times q} \\ V(\rho^p \circ \rho^q) &\to \rho^p \circ \rho^q . \end{aligned} \tag{3.71}$$

The fibres of $J_{lp\times qk}$ are given by $\mathrm{pr}^{-1}(\rho^p \circ \rho^q)$ and are isomorphic to $\mathbb{C}^{N^k_{lp\times q}}$. The two sets $\mathcal{N}^{\#}_{p\times q} = \mathrm{pr}^{-1}(\mathcal{M}^{\#}_{p\times q})$, $\# = I, II$ have a local product structure which is easily described in terms of the local product structure of J_{lpm}, J_{mqk}. Let $\{V^{lm}_\alpha(\rho^p_r)\}^{N^m_{lp}}_{\alpha=1}$ and $\{V^{mk}_\beta(\rho^q_r)\}^{N^k_{mq}}_{\beta=1}$ be orthonormal bases of $\mathscr{V}_l(\rho^p_r)_m$ and $\mathscr{V}_m(\rho^q_r)_k$, respectively. Then $\{V^{lm}_\alpha(\rho^p_r)V^{mk}_\beta(\rho^q_r)\}^{N^m_{lp},N^k_{mq}}_{\alpha,\beta=1}$ is an orthonormal basis in $\mathscr{V}_l(\rho^p_r \circ \rho^q_r)_k$ by Eq. (3.64). If $\rho^p \circ \rho^q$ is an element of $\mathcal{M}^{\#}_{p\times q}$, we obtain an orthonormal basis of intertwiners in $\mathscr{V}_l(\rho^p \circ \rho^q)_k$ by multiplying the orthonormal bases $\{V^{lm}_{\alpha\#}(\rho^p)\}^{N^m_{lp}}_{\alpha=1}$, $\{V^{mk}_{\beta\#}(\rho^q)\}^{N^k_{mq}}_{\beta=1}$ of $\mathscr{V}_l(\rho^p)_m$ and $\mathscr{V}_m(\rho^q)_k$, respectively, where $\rho^p \in \mathcal{M}^{\#}_p$, $\rho^q \in \mathcal{M}^{\#}_q$. The explicit choice of the operators $V^{lm}_{\alpha\#}(\rho^p)$, $V^{mk}_{\beta\#}(\rho^q)$ is given in Sec. 3.1, Eqs. (3.30)–(3.36). The local product structure of $\mathcal{N}^{\#}_{p\times q}$, $\# = I, II$ is then given by

$$\begin{aligned} \mathcal{N}^{\#}_{p\times q} &\to \mathcal{M}^{\#}_{p\times q} \times \mathbb{C}^{N^k_{lp\times q}} \\ V(\rho^p \circ \rho^q) &\to \left(\rho^p \circ \rho^q, \{\langle V^{lm}_{\alpha\#}(\rho^p)V^{mk}_{\beta\#}(\rho^q); V(\rho^p \circ \rho^q)\rangle\}^{N^m_{lp},N^k_{mq}}_{\alpha,\beta=1}\right). \end{aligned} \tag{3.72}$$

It is an easy task to calculate the transition functions of the bundle $J_{lp\times qk}$. If $\rho^p \circ \rho^q \in \mathcal{M}^{I}_{p\times q} \cap \mathcal{M}^{II}_{p\times q}$ and ρ^p, ρ^q are localized in cones $\mathscr{C}_p, \mathscr{C}_q$, respectively, then the equation

$$V^{lm}_{\alpha II}(\rho^p)V^{mk}_{\beta II}(\rho^q) = \sum_{n,\gamma,\delta} B_{lk}(II,I)^{n\gamma\delta}_{m\alpha\beta} V^{ln}_{\gamma I}(\rho^p)V^{nk}_{\delta I}(\rho^q) \tag{3.73}$$

defines a transition matrix $B_{lk}(II,I)$. Clearly

$$B_{lk}(II,I)^{n\gamma\delta}_{m\alpha\beta} = \delta^n_m B^m_{lk}(II,I)^{\gamma\delta}_{\alpha\beta}, \tag{3.74}$$

and on each subspace $\mathscr{V}_l(\rho^p)_m \otimes \mathscr{V}_m(\rho^q)_k$ of $\mathscr{V}_l(\rho^p \circ \rho^q)_k$, $B^m_{lk}(II,I)$ is given by

$$B^m_{lk}(II,I) = b_{lm}(II,I) \otimes b_{mk}(II,I), \tag{3.75}$$

so that, by Theorem 3.1,

$$B_{lk}^{m}(II, I) = \begin{cases} \mathbb{1} \otimes \mathbb{1} & \text{if } \mathscr{C}_p, \mathscr{C}_q \in \bar{\mathscr{P}}(\mathscr{C}_r) \\ e^{2\pi i(s_m - s_l)} \mathbb{1} \otimes \mathbb{1} & \text{if } \mathscr{C}_p \in \bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)}),\ \mathscr{C}_q \in \bar{\mathscr{P}}(\mathscr{C}_r) \\ e^{2\pi i(s_k - s_m)} \mathbb{1} \otimes \mathbb{1} & \text{if } \mathscr{C}_p \in \bar{\mathscr{P}}(\mathscr{C}_r),\ \mathscr{C}_q \in \bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)}) \\ e^{2\pi i(s_k - s_l)} \mathbb{1} \otimes \mathbb{1} & \text{if } \mathscr{C}_p, \mathscr{C}_q \in \bar{\mathscr{P}}(\mathscr{C}_{r(\pi,0)}) \end{cases} \tag{3.76}$$

where $\mathscr{C}_p$, $\mathscr{C}_q$ are the localization cones of ρ^p and ρ^q respectively.

4. Statistics and Fusion

In this section, we show that the field bundles $J_{l p \times q k}$, $J_{l q \times p k}$, $p, q, k, l \in L$ of a product representation give rise to a set of fusion and statistics matrices which obey properties analogous to those of the braiding and fusion matrices of conformal field theories [57].

4.1. *Statistics*

Let $p, q \in L$ be two irreducible representations and consider the field bundles $J_{l p \times q k}$, $J_{l q \times p k}$ of the product representations $p \times q$ and $q \times p$. As shown in Sec. 3, the coordinatization of these bundles requires the choice of a reference cone $\mathscr{C}_r$, two auxiliary cones $\mathscr{C}_a^{I}$, $\mathscr{C}_a^{II}$ and reference morphisms ρ_r^p, ρ_r^q localized in $\mathscr{C}_r$. Furthermore, orthonormal bases $\{V_\alpha^{lm}(\rho_r^p)\}_{\alpha=1}^{N_{lp}^m}$, $\{V_\beta^{mk}(\rho_r^q)\}_{\beta=1}^{N_{mq}^k}$, $\{V_\gamma^{ln}(\rho_r^q)\}_{\gamma=1}^{N_{lq}^n}$ and $\{V_\delta^{nk}(\rho_r^p)\}_{\delta=1}^{N_{np}^k}$ determine orthonormal bases $\{V_\alpha^{lm}(\rho_r^p) V_\beta^{mk}(\rho_r^q)\}_{\alpha,\beta=1}^{N_{lp}^m, N_{mq}^k}$ of $\mathscr{V}_l(\rho_r^p \circ \rho_r^q)_k$ and $\{V_\gamma^{ln}(\rho_r^q) V_\delta^{nk}(\rho_r^p)\}_{\gamma,\delta=1}^{N_{lq}^n, N_{np}^k}$ of $\mathscr{V}_l(\rho_r^q \circ \rho_r^p)_k$, as well as the local product structure of $\mathscr{N}_{p\times q}^{\#}$, $\mathscr{N}_{q\times p}^{\#}$, $\# = I, II$. If we choose $\rho^p \circ \rho^q \in \mathscr{M}_{p\times q}^{\#}$, $\# = I, II$, with ρ^p and ρ^q localized in spacelike separated cones (this will be denoted by $\rho^p \between \rho^q$ or $\mathscr{C}_p \between \mathscr{C}_q$ if $\mathscr{C}_p$ and $\mathscr{C}_q$ are the localization cones of ρ^p and ρ^q, respectively), then we see that

$$\{V_{\gamma\#}^{ln}(\rho^q) V_{\delta\#}^{nk}(\rho^p)\}_{\gamma,\delta=1}^{N_{lq}^n, N_{np}^k} \tag{4.1}$$

is an orthonormal basis of $\mathscr{V}_l(\rho^p \circ \rho^q)_k$. This follows directly from the fact that $\rho^p \circ \rho^q = \rho^q \circ \rho^p$ (Eq. (2.22)) and because

$$N_{l p\times q}^k = N_{l q\times p}^k = \sum_{n\in L} N_{lq}^n N_{np}^k \tag{4.2}$$

holds. The change of bases from $\{V_{\alpha\#}^{lm}(\rho^p) V_{\beta\#}^{mk}(\rho^q)\}_{\alpha,\beta=1}^{N_{lp}^m, N_{mq}^k}$ to $\{V_{\gamma\#}^{ln}(\rho^q) V_{\delta\#}^{nk}(\rho^p)\}_{\gamma,\delta=1}^{N_{lq}^n, N_{np}^k}$ is given by a unitary matrix $R_\#(l, \rho^p, \rho^q, k) \in U_{\mathbb{C}}(N_{l p\times q}^k)$, i.e.,

$$V_{\alpha\#}^{lm}(\rho^p) V_{\beta\#}^{mk}(\rho^q) = \sum_{n,\gamma,\delta} R_\#(l, \rho^p, \rho^q, k)_{m\alpha\beta}^{n\gamma\delta} V_{\gamma\#}^{ln}(\rho^q) V_{\delta\#}^{nk}(\rho^p) \tag{4.3}$$

is satisfied.

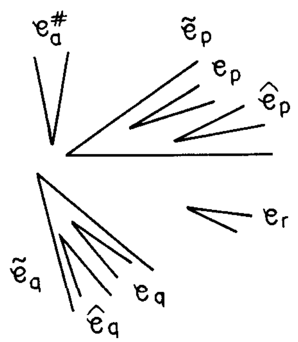


Fig. 4.1.

Lemma 4.1. *Let $\rho^p \in \mathscr{M}_p^{\#}$ and $\rho^q \in \mathscr{M}_q^{\#}$ be two morphisms with spacelike separated localization cones $\mathscr{C}_1$ and $\mathscr{C}_2$, $\# = I, II$. Then the matrices $R_{\#}(l, \rho^p, \rho^q, k)$ introduced in Eq. (4.3) only depend on the classes $\mathscr{M}_p^{\mathscr{C}_r}$, $\mathscr{M}_q^{\mathscr{C}_r}$ and on the relative asymptotic directions of ρ^p, ρ^q, that is*

$$R_{\#}(l, \rho^p, \rho^q, k) = \begin{cases} R^+(l, p, q, k) & \text{if } \mathrm{as}_{\#}\,\rho^p > \mathrm{as}_{\#}\,\rho^q \\ R^-(l, p, q, k) & \text{if } \mathrm{as}_{\#}\,\rho^p < \mathrm{as}_{\#}\,\rho^q . \end{cases} \tag{4.4}$$

□

Remark 4.2. We may rewrite Eq. (4.3) as follows:

$$V_\alpha^{lm}(\rho^p)V_\beta^{mk}(\rho^q) = \sum_{n,\gamma,\delta} R^{\pm}(l, p, q, k)_{m\alpha\beta}^{n\gamma\delta} V_\gamma^{ln}(\rho^q)V_\delta^{nk}(\rho^p) \tag{4.5}$$

for $\mathrm{as}\,\rho^p \gtrless \mathrm{as}\,\rho^q$. Unless necessary, we will suppress the index $\#$ specifying in which coordinate system the bases of $\mathscr{V}_l(\rho^p \circ \rho^q)_k$, $\mathscr{V}_l(\rho^q \circ \rho^p)_k$ are chosen (see Eq. (4.5)). □

Proof. The proof of Eq. (4.4) uses an argument invented in [17] and is essentially identical to the one given in [23]. Let ρ^p, ρ^q, $\hat{\rho}^p$, $\hat{\rho}^q$ be localized in $\mathscr{C}_p$, $\mathscr{C}_q$, $\hat{\mathscr{C}}_p$, $\hat{\mathscr{C}}_q$, respectively, $\rho^p \cong \hat{\rho}^p$, $\rho^q \cong \hat{\rho}^q$, so that $\mathscr{C}_p \cup \hat{\mathscr{C}}_p \subseteq \tilde{\mathscr{C}}_p$, $\mathscr{C}_q \cup \hat{\mathscr{C}}_q \subseteq \tilde{\mathscr{C}}_q$ hold, for two spacelike cones $\tilde{\mathscr{C}}_p$, $\tilde{\mathscr{C}}_q$ satisfying $\tilde{\mathscr{C}}_p \mathrel{)\!(} \tilde{\mathscr{C}}_q$ and $\tilde{\mathscr{C}}_p, \tilde{\mathscr{C}}_q \mathrel{)\!(} \mathscr{C}_a^{\#} + y$, for some $y \in \mathbb{M}^3$ (see Fig. 4.1).

Then we show that $R_{\#}(l, \rho^p, \rho^q, k)_{m\alpha\beta}^{n\gamma\delta}$ is constant under the change $\rho^p \to \hat{\rho}^p$, $\rho^q \to \hat{\rho}^q$. Repeated iteration of the preceding argument shows that

$$R_{\#}(l, \rho^p, \rho^q, k)_{m\alpha\beta}^{n\gamma\delta} = R_{\#}^{\pm}(l, p, q, k)_{m\alpha\beta}^{n\gamma\delta} \tag{4.6}$$

for $\mathrm{as}_{\#}\,\rho^p \gtrless \mathrm{as}_{\#}\,\rho^q$, respectively. Next, we check that

$$R_I^{\pm}(l, p, q, k)_{m\alpha\beta}^{n\gamma\delta} = R_{II}^{\pm}(l, p, q, k)_{m\alpha\beta}^{n\gamma\delta} \tag{4.7}$$

by choosing ρ^p, ρ^q in $\mathscr{M}_{p\times q}^I \cap \mathscr{M}_{p\times q}^{II}$ appropriately. This will complete the proof of Lemma 4.1.

Equation (4.3) implies that the matrix elements of $R_{\#}(l, \rho^p, \rho^q, k)$ are given by the scalar product

$$R_{\#}(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta} = \langle V^{ln}_{\gamma\#}(\rho^q) V^{nk}_{\delta\#}(\rho^p); V^{lm}_{\alpha\#}(\rho^p) V^{mk}_{\beta\#}(\rho^q)\rangle$$
$$= V^{nk}_{\delta\#}(\rho^p)^* V^{ln}_{\gamma\#}(\rho^q)^* V^{lm}_{\alpha\#}(\rho^p) V^{mk}_{\beta\#}(\rho^q). \tag{4.8}$$

The right hand side of Eq. (4.8) is a well-defined operator intertwining the representation k of the auxiliary algebra $\mathscr{B}^{\#}$ with itself, $\# = I, II$. Let $\hat{\Gamma}_p$, $\hat{\Gamma}_q$ be two unitary operators satisfying

$$\hat{\rho}^p(A)\hat{\Gamma}_p = \hat{\Gamma}_p \rho^p(A), \qquad \hat{\rho}^q(A)\hat{\Gamma}_q = \hat{\Gamma}_q \rho^q(A) \tag{4.9}$$

and

$$V^{lm}_{\alpha\#}(\hat{\rho}^p) = l(\hat{\Gamma}_p) V^{lm}_{\alpha\#}(\rho^p), \qquad V^{mk}_{\beta\#}(\hat{\rho}^q) = m(\hat{\Gamma}_q) V^{mk}_{\beta\#}(\rho^q)$$
$$V^{ln}_{\gamma\#}(\hat{\rho}^q) = l(\hat{\Gamma}_q) V^{ln}_{\gamma\#}(\rho^q), \qquad V^{nk}_{\delta\#}(\hat{\rho}^p) = n(\hat{\Gamma}_p) V^{nk}_{\delta\#}(\rho^p). \tag{4.10}$$

The localization properties of the morphisms imply that

$$\rho^q(\hat{\Gamma}^*_p) = \hat{\Gamma}^*_p, \; \rho^p(\hat{\Gamma}_q) = \hat{\Gamma}_q,$$
$$\hat{\Gamma}_p \hat{\Gamma}_q = \hat{\Gamma}_q \hat{\Gamma}_p. \tag{4.11}$$

Let us now compute $R_{\#}(l, \hat{\rho}^p, \hat{\rho}^q, k)^{n\gamma\delta}_{m\alpha\beta}$

$$R_{\#}(l, \hat{\rho}^p, \hat{\rho}^q, k)^{n\gamma\delta}_{m\alpha\beta} = V^{nk}_{\delta\#}(\hat{\rho}^p)^* V^{ln}_{\gamma\#}(\hat{\rho}^q)^* V^{lm}_{\alpha\#}(\hat{\rho}^p) V^{mk}_{\beta\#}(\hat{\rho}^q)$$
$$= V^{nk}_{\delta\#}(\rho^p)^* n(\hat{\Gamma}^*_p) V^{ln}_{\gamma\#}(\rho^q)^* l(\hat{\Gamma}^*_q) l(\hat{\Gamma}_p) V^{lm}_{\alpha\#}(\rho^p)$$
$$\times\, m(\hat{\Gamma}_q) V^{mk}_{\beta}(\rho^q) \tag{4.12}$$

by (4.10). Using the intertwining properties we may rewrite (4.12) as

$$R_{\#}(l, \hat{\rho}^p, \hat{\rho}^q, k)^{n\gamma\delta}_{m\alpha\beta} = V^{nk}_{\delta\#}(\rho^p)^* V^{ln}_{\gamma\#}(\rho^q)^* l(\rho^q(\hat{\Gamma}^*_p)\hat{\Gamma}^*_q)$$
$$\times\, l(\hat{\Gamma}_p \rho^p(\hat{\Gamma}_q)) V^{lm}_{\alpha\#}(\rho^p) V^{mk}_{\beta\#}(\rho^q). \tag{4.13}$$

But (4.11) and (4.8) imply that

$$R_{\#}(l, \hat{\rho}^p, \hat{\rho}^q, k)^{n\gamma\delta}_{m\alpha\beta} = R_{\#}(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta}. \tag{4.14}$$

This completes the first part of the proof. The second part is an immediate consequence of Eq. (3.76). Since

$$R_{\#}(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta} = \langle V^{ln}_{\gamma\#}(\rho^q) V^{nk}_{\delta\#}(\rho^p);\ V^{lm}_{\alpha\#}(\rho^p) V^{mk}_{\beta\#}(\rho^q)\rangle, \tag{4.15}$$

$\# = I, II$, we may express $R_{II}(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta}$ in terms of $R_I(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta}$:

$$\begin{aligned} R_{II}(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta} &= \langle V^{ln}_{\gamma II}(\rho^q) V^{nk}_{\delta II}(\rho^p);\ V^{lm}_{\alpha II}(\rho^p) V^{mk}_{\beta II}(\rho^q)\rangle \\ &= \langle \tilde{B}^n_{lk}(II, I) V^{ln}_{\gamma I}(\rho^q) V^{nk}_{\delta I}(\rho^p);\ B^m_{lk}(II, I) V^{lm}_{\alpha I}(\rho^p) V^{mk}_{\beta I}(\rho^q)\rangle, \end{aligned} \tag{4.16}$$

where $B^m_{lk}(II, I)$ are the transition matrices of $J_{lp\times qk}$ and $\tilde{B}^n_{lk}(II, I)$ the ones of $J_{lq\times pk}$. Taking advantage of the fact that they reduce to phase factors (see Eq. (3.76)), we may write

$$R_{II}(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta} = \tilde{B}^n_{lk}(II, I)^* B^m_{lk}(II, I) \cdot R_I(l, \rho^p, \rho^q, k)^{n\gamma\delta}_{m\alpha\beta}. \tag{4.17}$$

Using Eq. (3.76) in the two situations shown in Fig. 4.2,

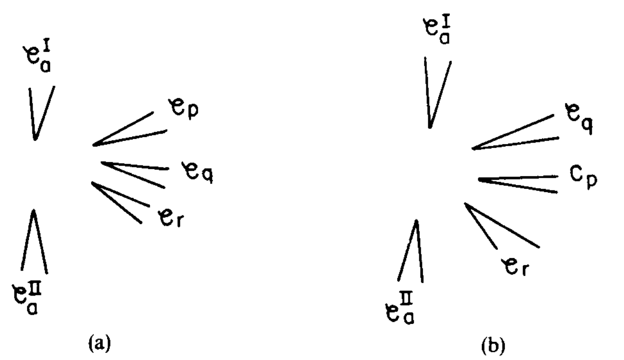


Fig. 4.2.

we find that

$$R^+_I(l, p, q, k)^{n\gamma\delta}_{m\alpha\beta} = R^+_{II}(l, p, q, k)^{n\gamma\delta}_{m\alpha\beta} \tag{4.18}$$

(see Fig. 4.2(a)) and

$$R^-_I(l, p, q, k)^{n\gamma\delta}_{m\alpha\beta} = R^-_{II}(l, p, q, k)^{n\gamma\delta}_{m\alpha\beta} \tag{4.19}$$

(see Fig. 4.2(b)). This completes the proof of the lemma. ∎

In the next subsection, we will make further use of Eq. (4.17) to derive an identity between the R^+ and R^- matrices.

Next we relate our formalism to the one of Doplicher, Haag, and Roberts [17].

Lemma 4.3. *There exist two unitary operators $\varepsilon^{\pm}_{\rho^p,\rho^q}$ in the algebras $\mathscr{B}^{\#}$, $\# = I, II$ which intertwine the representations $\rho^p \circ \rho^q$ and $\rho^q \circ \rho^p$ of the observable algebra $\mathscr{A}$ such that*

$$l(\varepsilon^{\pm}_{\rho^p,\rho^q})V^{lm}_{\alpha}(\rho^p)V^{mk}_{\beta}(\rho^q) = \sum_{n,\gamma,\delta} R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} V^{ln}_{\gamma}(\rho^q)V^{nk}_{\delta}(\rho^p). \tag{4.20}$$

The statistics matrices $R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta}$ are obtained by expressing $l(\varepsilon^{\pm}_{\rho^p,\rho^q})$ in specific bases of $\mathscr{V}_l(\rho^p \circ \rho^q)_k$ and $\mathscr{V}_l(\rho^q \circ \rho^p)_k$:

$$R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} = V^{nk}_{\delta}(\rho^p)^* V^{ln}_{\gamma}(\rho^q)^* l(\varepsilon^{\pm}_{\rho^p,\rho^q}) V^{lm}_{\alpha}(\rho^p)V^{mk}_{\beta}(\rho^q) \tag{4.21}$$

and conversely, one recovers the statistics operators $l(\varepsilon^{\pm}_{\rho^p,\rho^q})$ from

$$l(\varepsilon^{\pm}_{\rho^p,\rho^q}) = \sum_{\substack{n,\gamma,\delta\\ k,m,\alpha,\beta}} R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} V^{ln}_{\gamma}(\rho^q)V^{nk}_{\delta}(\rho^p)V^{mk}_{\beta}(\rho^q)^* V^{lm}_{\alpha}(\rho^p)^*. \tag{4.22}$$

□

Proof. The proof consists in a reconstruction of the statistics operators, as originally proposed by Doplicher, Haag and Roberts in [17]. For $\rho^p \circ \rho^q \in \mathscr{M}^{\#}_{p\times q}$, choose two spacelike separated morphisms $\hat{\rho}^p$, $\hat{\rho}^q$ such that $\hat{\rho}^p \circ \hat{\rho}^q \in \mathscr{M}^{\#}_{p\times q}$ and intertwiners $\hat{\Gamma}_p, \hat{\Gamma}_q \in \mathscr{U}^{\#}$ satisfying Eq. (4.9), (4.10), $\# = I$ or II. Then

$$V^{lm}_{\alpha}(\rho^p)V^{mk}_{\beta}(\rho^q) = l\big(\rho^p(\hat{\Gamma}^*_q)\hat{\Gamma}^*_p\big)V^{lm}_{\alpha}(\hat{\rho}^p)V^{mk}_{\beta}(\hat{\rho}^q). \tag{4.23}$$

Since $\hat{\rho}^p$ and $\hat{\rho}^q$ are spacelike separated, we may apply Lemma 4.1, i.e.,

$$V^{lm}_{\alpha}(\rho^p)V^{mk}_{\beta}(\rho^q) = l\big(\rho^p(\hat{\Gamma}^*_q)\hat{\Gamma}^*_p\big) \sum_{n,\gamma,\delta} R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} V^{ln}_{\gamma}(\hat{\rho}^q)V^{nk}_{\delta}(\hat{\rho}^p) \tag{4.24}$$

where the $\pm$ sign depends on whether as $\hat{\rho}^p \gtrless$ as $\hat{\rho}^q$ holds. We use Eq. (4.10) again to return to the original intertwiners:

$$\begin{aligned} V^{lm}_{\alpha}(\rho^p)V^{mk}_{\beta}(\rho^q) &= l\big(\rho^p(\hat{\Gamma}^*_q)\hat{\Gamma}^*_p\big)l\big(\hat{\Gamma}_q\rho^q(\hat{\Gamma}_p)\big) \\ &\quad\times \sum_{n,\gamma,\delta} R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} V^{ln}_{\gamma}(\rho^q)V^{nk}_{\delta}(\rho^p). \end{aligned} \tag{4.25}$$

This may be rewritten as

$$\begin{aligned} &l\big(\rho^q(\hat{\Gamma}^*_p)\hat{\Gamma}^*_q\hat{\Gamma}_p\rho^p(\hat{\Gamma}_q)\big)V^{lm}_{\alpha}(\rho^p)V^{mk}_{\beta}(\rho^q) \\ &\quad = \sum_{n,\gamma,\delta} R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} V^{ln}_{\gamma}(\rho^q)V^{nk}_{\delta}(\rho^p). \end{aligned} \tag{4.26}$$

We define the statistics operators by the equation

$$\rho^q(\hat{\Gamma}_p^*)\hat{\Gamma}_q^*\hat{\Gamma}_p\rho^p(\hat{\Gamma}_q) = \varepsilon^{\pm}_{\rho^p,\rho^q}, \tag{4.27}$$

where the $\pm$ sign depends again on whether as $\hat{\rho}^p \gtrless$ as $\hat{\rho}^q$. Hence, we may rewrite Eq. (4.26) as

$$l(\varepsilon^{\pm}_{\rho^p,\rho^q})V^{lm}_\alpha(\rho^p)V^{mk}_\beta(\rho^q) = \sum_{n,\gamma,\delta} R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta}V^{ln}_\gamma(\rho^q)V^{nk}_\delta(\rho^p) \tag{4.28}$$

which is (4.20). Equations (4.21) and (4.22) are merely a rewriting of (4.20) and are easily obtained by using the orthogonality and completeness relations for the bases of $\mathscr{V}_l(\rho^p \circ \rho^q)_k$, $\mathscr{V}_l(\rho^q \circ \rho^p)_k$. ∎

Our discussion is summarized in the following theorem.

Theorem 4.4. *The mappings*

$$\begin{gathered}\rho^p \circ \rho^q \in \mathscr{M}^{\mathscr{C}_r}_{p\times q} \to \rho^q \circ \rho^p \in \mathscr{M}^{\mathscr{C}_r}_{q\times p}\\ V \in \mathscr{V}_l(\rho^p \circ \rho^q)_k \to l(\varepsilon^{\pm}_{\rho^p,\rho^q})V \in \mathscr{V}_l(\rho^q \circ \rho^p)_k\end{gathered} \tag{4.29}$$

define two bijections

$$R^{\pm}_{p,q} : J_{lp\times qk} \to J_{lq\times pk} \tag{4.30}$$

which preserve the fibres of $J_{lp\times qk}$. *Their inverses,*

$$R^{\pm}_{q,p} : J_{lq\times pk} \to J_{lp\times qk} \tag{4.31}$$

are given by

$$\begin{gathered}\rho^q \circ \rho^p \in \mathscr{M}^{\mathscr{C}_r}_{q\times p} \to \rho^p \circ \rho^q \in \mathscr{M}^{\mathscr{C}_r}_{p\times q}\\ V \in \mathscr{V}_l(\rho^q \circ \rho^p) \to l(\varepsilon^{\pm}_{\rho^q,\rho^p})V \in \mathscr{V}_l(\rho^p \circ \rho^q)_k\,.\end{gathered} \tag{4.32}$$

Furthermore, if we choose local coordinates for the bundles $J_{lp\times qk}$ *and* $J_{lq\times pk}$, *as specified above, the isomorphisms* $R^{\pm}_{p,q}$ *are determined by the matrices* $R^{\pm}(l,p,q,k) \in U_{\mathbb{C}}(N^k_{lp\times q})$. □

The relation

$$R^{\mp}_{q,p} \circ R^{\pm}_{p,q} = id|_{J_{lp\times qk}} \tag{4.33}$$

will be proven in the next subsection.

Finally, we remark that the $R^{\pm}$-matrix elements depend on the choice of the orthonormal bases in $\mathscr{V}_l(\rho^p_r)_m$, $\mathscr{V}_m(\rho^q_r)_k$, $\mathscr{V}_l(\rho^q_r)_n$ and $\mathscr{V}_n(\rho^p_r)_k$ (see Eq. (4.5)). If we apply the unitary "gauge" transformations

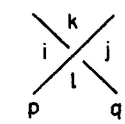
$$
\begin{aligned}
U(l,p,m) &= \left(U(l,p,m)^{\alpha}_{\bar\alpha}\right) \in U_{\mathbb{C}}(N^{m}_{lp}) \\
U(m,q,k) &= \left(U(m,q,k)^{\beta}_{\bar\beta}\right) \in U_{\mathbb{C}}(N^{k}_{mq}) \\
U(l,q,n) &= \left(U(l,q,n)^{\gamma}_{\bar\gamma}\right) \in U_{\mathbb{C}}(N^{n}_{lq}) \\
U(n,p,k) &= \left(U(n,p,k)^{\delta}_{\bar\delta}\right) \in U_{\mathbb{C}}(N^{k}_{np})
\end{aligned}
\tag{4.34}
$$

on these four vector spaces then the $R^{\pm}$-matrices transform as follows: $R^{\pm} \to R^{\pm}_{U}$, where $R^{\pm}_{U}$ is given by

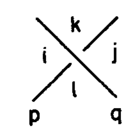
$$
\begin{aligned}
R^{\pm}_{U}(l,p,q,k)^{n\bar\gamma\bar\delta}_{m\bar\alpha\bar\beta} = \sum_{\alpha,\beta,\gamma,\delta} & \overline{U(l,q,n)^{\bar\gamma}_{\gamma}}\,\overline{U(n,p,k)^{\bar\delta}_{\delta}}\, R^{\pm}(l,p,q,k)^{n\gamma\delta}_{m\alpha\beta} \\
& \times U(l,p,m)^{\alpha}_{\bar\alpha}\, U(m,q,k)^{\beta}_{\bar\beta}\,.
\end{aligned}
\tag{4.35}
$$

4.2. *Properties of the statistics matrices*

In this subsection we derive some basic properties of the statistics matrices. The following graphical notation is convenient.

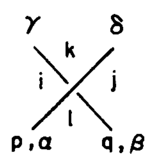


$$\leftrightarrow R^{+}(i,p,q,j)^{k}_{l} \tag{4.36}$$



$$\leftrightarrow R^{-}(i,p,q,j)^{k}_{l} \tag{4.37}$$

In Eqs. (4.36), (4.37) we have dropped the Greek multiplicity indices α, β, γ, δ of the R-matrices; the complete notation would be



$$\leftrightarrow R^{+}(i,p,q,j)^{k\gamma\delta}_{l\alpha\beta} \tag{4.38}$$

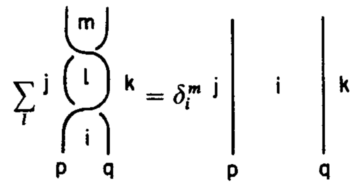
Equations between R-matrices may be written, in this notation, by composing the diagrams introduced above. Composition is defined by contracting strings in the following way: we read the complete diagram from bottom to top; Greek indices at the tips of contracted strings and Latin indices enclosed by strings in bounded regions are to be summed over. The latter sums will usually be written explicitly in this section. The simplest equation between R^{+} and R^{-} is obtained by iterating Eq. (4.5):

$$\sum_{l,\gamma,\delta} R^+(j,p,q,k)^{l\gamma\delta}_{i\alpha\beta} R^-(j,q,p,k)^{m\rho\eta}_{l\gamma\delta} = \delta^m_i \delta^\rho_\alpha \delta^\eta_\beta \,. \tag{4.39}$$

Of course, unitarity of the $R^\pm$ matrices combined with this equation, implies that

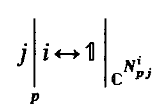
$$R^-(j,q,p,k)^{m\rho\eta}_{l\gamma\delta} = \overline{R^+(j,p,q,k)^{l\gamma\delta}_{m\rho\eta}} \,. \tag{4.40}$$

Graphically, Eq. (4.39) is given by



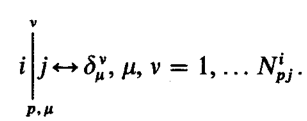
(4.41)

where we used the following additional notation:



(4.42)

or equivalently,



(4.43)

To derive Eq. (4.39), one considers two spacelike separated morphisms ρ^p, ρ^q with as $\rho^p >$ as ρ^q and permutes the order of the factors in $V^{lm}_\alpha(\rho^p)V^{mk}_\beta(\rho^q)$ twice, each time using Eq. (4.5).

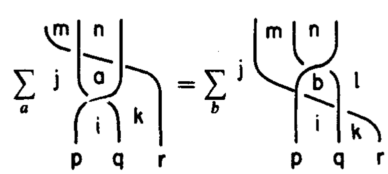
In the same way, by considering three spacelike separated morphisms ρ^p, ρ^q, ρ^r such that as $\rho^p >$ as $\rho^q >$ as ρ^r and permuting the order of

$$V^{ji}_\mu(\rho^p)V^{ik}_\nu(\rho^q)V^{kl}_\kappa(\rho^r) \tag{4.44}$$

to

$$V^{jm}_\alpha(\rho^r)V^{mn}_\beta(\rho^q)V^{nl}_\gamma(\rho^p) \tag{4.45}$$

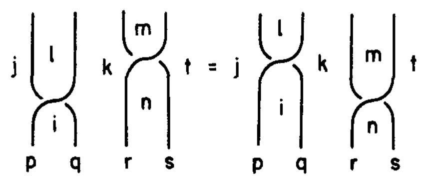
in two distinct ways, we obtain



(4.46)

Further related identities follow by considering a different ordering of the asymptotic directions of ρ^p, ρ^q, ρ^r. The Eqs. (4.46) are homogeneous cubic equations in the matrices $R^{\pm}$. They represent the sos-form of the Yang-Baxter equations in the special case where the spectral parameter has only two values, $+$ and $-$. The original derivation of (4.46) from (4.44) and (4.45) may be found in [26].

From Eq. (4.46) and



(4.47)

we conclude that the matrices $(R^{\pm}(j, p, q, k)^l_m)$ generate a unitary representation of the groupoid B^c_n of coloured braids on n strands (see Sec. 6). It is easy, albeit somewhat lengthy, to check that Eq. (4.46) and (4.47) are equivalent to

$$j\big(\varepsilon^+_{\rho^q,\rho^r}\rho^q(\varepsilon^+_{\rho^p,\rho^r})\varepsilon^+_{\rho^p,\rho^q}\big) = j\big(\rho^r(\varepsilon^+_{\rho^p,\rho^q})\varepsilon^+_{\rho^p,\rho^r}\rho^p(\varepsilon^+_{\rho^q,\rho^r})\big) \tag{4.48}$$

and

$$j\big(\rho^q \circ \rho^p(\varepsilon^+_{\rho^r,\rho^s})\varepsilon^+_{\rho^p,\rho^q}\big) = j(\varepsilon^+_{\rho^p,\rho^q}\rho^p \circ \rho^q(\varepsilon^+_{\rho^r,\rho^s})) \tag{4.49}$$

in the algebraic formalism; see [22, 23].

Next, we derive a basic relation between R^+ and R^-.

Lemma 4.5. *The matrix elements $R^{\pm}(l, p, q, k)^i_j$ satisfy the identity*

$$R^+(l, p, q, k)^i_j = e^{2\pi i(s_j + s_i - s_k - s_l)} R^-(l, p, q, k)^i_j. \tag{4.50}$$

□

This identity is a well-known identity in conformal field theory [49] if one reinterprets s_j as the conformal dimension of a representation j of some chiral algebra $\mathscr{A}$. The proof follows from Eqs. (4.17) and (3.76) by analyzing the situation shown in Fig. 4.3 below.

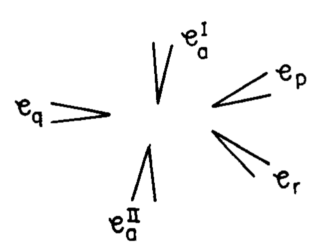


Fig. 4.3.

More details are contained in [24].

A consequence of Eq. (4.50) is that if all representations $j \in L$ have integer spin, i.e.,

$$s_j = 0\,(\mathrm{mod}\,\mathbb{Z}), \qquad \text{for all } j \in L$$

then

$$R^+(i,p,q,k)^l_j = R^-(i,p,q,k)^l_j \tag{4.51}$$

holds for arbitrary i, p, q, k, j and l in L. In this case, Eqs. (4.46) and (4.47) imply that the matrices $(R(i,p,q,k)^l_j)$ define representations of the permutation groups S_n, of n elements. Hence, in a theory in which all representations have integer spin, the statistics of the intertwiners $\{V^{ij}_\alpha(\rho^p)\}^{N^j_{ip}}_{\alpha=1}$ reduces to the ordinary permutation group statistics, as analyzed by Doplicher, Haag and Roberts in [17].

Lemma 4.6. *The following equations hold.*

$$\text{(i)}\ \ R^\pm(j,p,q,1)^{l\alpha\beta}_{k\mu\nu} = \delta^{\bar q}_k \delta^l_{\bar p} \delta^1_\nu \delta^\beta_1 R^\pm(j,p,q,1)^{\bar p\alpha 1}_{\bar q\mu 1} \tag{4.52}$$

$$\text{(ii)}\ \ R^\pm(1,p,q,j)^{l\alpha\beta}_{k\mu\nu} = \delta^p_k \delta^l_q \delta^1_\mu \delta^\alpha_1 R^\pm(1,p,q,j)^{q1\beta}_{p1\nu}. \tag{4.53}$$

Furthermore, it is always possible to choose coordinates on the bundles $J_{l\bar q k}$, $J_{k\bar p j}$, $J_{l\bar p m}$ and $J_{m\bar q j}$ so that

$$\text{(iii)}\ \ R^-(l,\bar q,\bar p,j)^{m\delta\gamma}_{k\beta\alpha} = \overline{R^+(j,p,q,l)^{m\gamma\delta}_{k\alpha\beta}}. \tag{4.54}$$

□

Equations (4.52) and (4.53) follow from the fact that the intertwiner spaces $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_p(\bar\rho^p)_1$ are one-dimensional and the proof of Eq. (4.54) will be given in Sec. 5. Lemma 4.6 permits us to derive a connection between spin and statistics of sectors. The proof of this connection only uses Eq. (4.50) and Lemma 4.6 (iii), but does not require Lorentz covariance of the theory. It may therefore be valid in certain non-relativistic theories as well.

In order to derive our connection between spin and statistics, we first note that, by parts (i) and (ii) of Lemma 4.6, the only non-vanishing elements of the matrices $(R^\pm(1,p,\bar p,1)^{m\alpha\beta}_{k\mu\nu})$ are $R^\pm(1,p,\bar p,1)^{\bar p 11}_{p11}$. By (iii) and since $\bar{\bar p} = p$,

$$R^-(1,p,\bar p,1)^{\bar p 11}_{p11} = \overline{R^+(1,p,\bar p,1)^{\bar p 11}_{p11}}. \tag{4.55}$$

Taking into account the fact that the matrix $(R^+(1,p,\bar p,1)^{\bar p 11}_{p11})$ is unitary, we may introduce the following notation:

$$R^+(1,p,\bar p,1)^{\bar p 11}_{p11} = e^{2\pi i\theta_{p,\bar p}} \tag{4.56}$$

and by (4.55)

$$R^-(1,p,\bar{p},1)^{\bar{p}11}_{p11} = e^{-2\pi i\theta_{p,\bar{p}}}. \tag{4.57}$$

Equation (4.39) means that

$$\theta_{p,\bar{p}} = \theta_{\bar{p},p} \,(\mathrm{mod}\,\mathbb{Z}). \tag{4.58}$$

Next, we apply the fundamental relation (4.50) to conclude that

$$e^{2\pi i\theta_{p,\bar{p}}} = e^{2\pi i(s_p+s_{\bar{p}})}e^{-2\pi i\theta_{p,\bar{p}}}, \tag{4.59}$$

where we have used that $s_1 = 0\,(\mathrm{mod}\,\mathbb{Z})$. Equation (4.59) means that

$$\theta_{p,\bar{p}} = \frac{1}{2}(s_p + s_{\bar{p}})\left(\mathrm{mod}\,\frac{1}{2}\mathbb{Z}\right); \tag{4.60}$$

this is the desired spin-statistics connection. In Sec. 5 we will define the statistics parameter λ_p of a sector $p \in L$ [17, 18] and show that, for a certain choice of bases in the intertwiner spaces, $\theta_{p,\bar{p}}$ coincides with the phase of λ_p:

$$\lambda_p = |\lambda_p| e^{-2\pi i\theta_{p,\bar{p}}}. \tag{4.61}$$

Furthermore, it follows from Eq. (4.59) that if

$$s_p = \theta_{p,\bar{p}}\,(\mathrm{mod}\,\mathbb{Z}) \tag{4.62}$$

then

$$s_{\bar{p}} = s_p\,(\mathrm{mod}\,\mathbb{Z}) \tag{4.63}$$

holds. It will be convenient for later results to assume that Eq. (4.62) and consequently (4.63) are valid. (Lorentz covariance and an assumption of strict localizability of charged fields on spacelike strings actually permits us to sharpen our connection between spin and statistics and prove Eq. (4.62).)

4.3. *Fusion of intertwiners*

From the decomposition of a product

$$i \times j = \bigoplus_{l\in L} \bigoplus_{\mu=1}^{N^l_{ij}} l^{(\mu)} \tag{4.64}$$

for $i, j \in L$ and from the fact that

$$s \times (i \times j) = (s \times i) \times j \tag{4.65}$$

holds, we now derive the existence of unitary matrices governing the expansion of intertwiners $V_\alpha^{nk}(\rho^i)V_\beta^{km}(\rho^j) \in J_{ni\times jm}$ in terms of intertwiners $V_\gamma^{nm}(\rho^l) \in J_{nlm}$. Consider two representations i, $j \in L$, whose product $i \times j$ decomposes as in Eq. (4.64). We assume that a reference cone $\mathscr{C}_r$, as well as two auxiliary cones $\mathscr{C}_a^I$ and $\mathscr{C}_a^{II}$ have been chosen such that $\mathscr{M}_i^{\mathscr{C}_r}$, $\mathscr{M}_{i\times j}^{\mathscr{C}_r}$ and field bundles can be defined as discussed in the preceding sections. If ρ_r^i, ρ_r^j and ρ_r^l are morphisms belonging to the classes i, j and $l \in L$, respectively, and localized in the interior of the cone $\mathscr{C}_r$, then Eq. (4.64) implies the existence of partial isometries $\Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\mu)$ in $\mathscr{A}(\mathscr{C}_r)^{-w}\mu = 1, \ldots N_{ij}^l$ intertwining the representations $\rho_r^i \circ \rho_r^j$ of $\mathscr{A}$ with ρ_r^l:

$$\rho_r^i \circ \rho_r^j(A)\Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\mu) = \Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\mu)\rho_r^l(A). \tag{4.66}$$

This set of intertwiners is a complex vector space of dimension N_{ij}^l which we denote by $\mathscr{V}(\rho_r^i \circ \rho_r^j;\rho_r^l)$. We choose the $\{\Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\mu)\}$ to be an orthonormal basis in $\mathscr{V}(\rho_r^i \circ \rho_r^j;\rho_r^l)$ with respect to the obvious scalar product. If $J_{ni\times jm}$ is an intertwiner bundle of the representation $i \times j$ and $\{V_\alpha^{nk}(\rho_r^i)V_\beta^{km}(\rho_r^j)\}_{\alpha,\beta=1}^{N_{ni}^k,N_{kj}^m}$ an orthonormal basis of the fibre $\mathscr{V}_n(\rho_r^i \circ \rho_r^j)_m$, then the decomposition (4.64) along with

$$N_{ni\times j}^m = \sum_{l\in L} N_{ij}^l N_{nl}^m \tag{4.67}$$

implies that

$$\{n(\Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\mu))V_\nu^{nm}(\rho_r^l)\}_{\mu,\nu=1}^{N_{ij}^l,N_{nl}^m} \subseteq \mathscr{V}_n(\rho_r^i \circ \rho_r^j)_m \tag{4.68}$$

forms an orthonormal basis of $\mathscr{V}_n(\rho_r^i \circ \rho_r^j)_m$. Hence there are unitary matrices $\hat{F}(n,i,j,m)$ such that

$$V_\alpha^{nk}(\rho_r^i)V_\beta^{km}(\rho_r^j) = \sum_{l,\gamma,\delta} \hat{F}(n,i,j,m)_{k\alpha\beta}^{l\gamma\delta} n(\Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\gamma))V_\delta^{nm}(\rho_r^l). \tag{4.69}$$

Expansion (4.69) is called fusion, the matrices $\hat{F}(n,i,j,m)$ are called fusion matrices. We now show how to compute the fusion matrices $\hat{F}$ in terms of the statistics matrices.

Let us start with the special case of fusion on the vacuum sector. We consider the field bundle $J_{\bar{l}i\times j1}$ of the product representation $i \times j$ and pick the fibre $\mathscr{V}_{\bar{l}}(\rho_r^i \circ \rho_r^j)_1$ over the reference morphism $\rho_r^i \circ \rho_r^j$. The usual orthonormal basis of $\mathscr{V}_{\bar{l}}(\rho_r^i \circ \rho_r^j)_1$ is

$$\{V_\alpha^{\bar{l}\bar{j}}(\rho_r^i)V^{\bar{j}1}(\rho_r^j)\}_{\alpha=1}^{N_{\bar{l}i}^{\bar{j}}}, \tag{4.70}$$

whereas

$$\{\bar{l}(\Gamma_{\rho_r^i\circ\rho_r^j,\rho_r^l}(\mu))V^{\bar{l}1}(\rho_r^l)\}_{\mu=1}^{N_{ij}^l} \tag{4.71}$$

is the analogue of (4.68) in this simple case. Clearly, $N_{\bar{l}i}^{\bar{j}} = N_{ij}^l$. By performing a unitary "gauge" transformation in the intertwiner space $\mathscr{V}(\rho_r^i \circ \rho_r^j; \rho_r^l)$ (see Eqs. (4.34) and (4.35)), we can redefine the intertwiners $\Gamma_{\rho_r^i \circ \rho_r^j, \rho_r^l}(\mu)$ so that

$$\langle V_\alpha^{\bar{1}\bar{j}}(\rho_r^i) V^{\bar{j}1}(\rho_r^j); \bar{l}(\Gamma_{\rho_r^i \circ \rho_r^j, \rho_r^l}(\mu)) V^{\bar{l}1}(\rho_r^l)\rangle = \delta_{\alpha\mu} \tag{4.72}$$

holds. From now on, we assume that the intertwiners $\{\Gamma_{\rho_r^i \circ \rho_r^j, \rho_r^l}(\mu)\}$ in $\mathscr{V}(\rho_r^i \circ \rho_r^j, \rho_r^l)$ are normalized as in (4.72). We generalize this fusion identity to arbitrary $\rho^i \circ \rho^j \in \mathscr{M}_{i \times j}^\#$, $\rho^l \in \mathscr{M}_l^\#$, $\# = I, II$ by moving an intertwiner $V_{\alpha\#}^{\bar{1}\bar{j}}(\rho^i) V_\#^{\bar{j}1}(\rho^j)$ back to the fibre $\mathscr{V}_{\bar{l}}(\rho_r^i \circ \rho_r^j)_1$ before fusing.

$$\begin{aligned} V_{\mu\#}^{\bar{1}\bar{j}}(\rho^i) V_\#^{\bar{j}1}(\rho^j) &= \bar{l}(\rho^i(\Gamma^\#_{\rho^j \rho_r^j})\Gamma^\#_{\rho^i \rho_r^i}) V_{\alpha\#}^{\bar{1}\bar{j}}(\rho_r^i) V_\#^{\bar{j}1}(\rho_r^j) \\ &= \bar{l}(\rho^i(\Gamma^\#_{\rho^j \rho_r^j})\Gamma^\#_{\rho^i \rho_r^i} \Gamma_{\rho_r^i \circ \rho_r^j, \rho_r^l}(\mu)) V_\#^{\bar{l}1}(\rho_r^l) \\ &= \bar{l}(\rho^i(\Gamma^\#_{\rho^j \rho_r^j})\Gamma^\#_{\rho^i \rho_r^i} \Gamma_{\rho_r^i \circ \rho_r^j, \rho_r^l}(\mu) \Gamma^{\#*}_{\rho^l \rho_r^l}) V_\#^{\bar{l}1}(\rho^l) \end{aligned} \tag{4.73}$$

where the unitaries $\Gamma^\#_{\rho^i \rho_r^i}$, $\Gamma^\#_{\rho^j \rho_r^j}$ and $\Gamma^\#_{\rho^l \rho_r^l}$ have been defined in Sec. 3.1. If we set

$$\Gamma^\#_{\rho^i \circ \rho^j, \rho^l}(\mu) = \rho^i(\Gamma^\#_{\rho^j \rho_r^j})\Gamma^\#_{\rho^i \rho_r^i} \Gamma_{\rho_r^i \circ \rho_r^j, \rho_r^l}(\mu) \Gamma^{\#*}_{\rho^l \rho_r^l}, \tag{4.74}$$

$\# = I, II$, we can rewrite the fusion identity (4.73) in compact form:

$$V_{\mu\#}^{\bar{1}\bar{j}}(\rho^i) V_\#^{\bar{j}1}(\rho^j) = \bar{l}(\Gamma^\#_{\rho^i \circ \rho^j, \rho^l}(\mu)) V_\#^{\bar{l}1}(\rho^l), \qquad \# = I, II. \tag{4.75}$$

If $\mathscr{S}$ is any simple domain containing the localization cones of ρ^i, ρ^j and ρ^l in its interior, it follows that $\Gamma^\#_{\rho^i \circ \rho^j, \rho^l}(\mu) \in \mathscr{A}(\mathscr{S})^{-w}$.

After these preliminaries, we are ready to state our main result on fusion.

Theorem 4.7. *Let $i, j \in L$ be two irreducible representations of $\mathscr{A}$ which satisfy*

$$i \times j \cong \bigoplus_{l \in L} \bigoplus_{\mu=1}^{N_{ij}^l} l^{(\mu)}. \tag{4.76}$$

If ρ^i, ρ^j, ρ^l are morphisms belonging to $\mathscr{M}_i^\#$, $\mathscr{M}_j^\#$ and $\mathscr{M}_l^\#$ respectively, $\# = I, II$, then there exist unitary matrices $(\hat{F}(n,i,j,m)_{k\alpha\beta}^{l\gamma\delta})$, depending only on the representations m, i, j, n, l and k, such that

$$V_{\alpha\#}^{nk}(\rho^i) V_{\beta\#}^{km}(\rho^j) = \sum_{l,\gamma,\delta} \hat{F}(n,i,j,m)_{k\alpha\beta}^{l\gamma\delta} n(\Gamma^\#_{\rho^i \circ \rho^j, \rho^l}(\gamma)) V_{\delta\#}^{nm}(\rho^l) \tag{4.77}$$

holds, $\# = I, II$. The fusion matrices $(\hat{F}(n,i,j,m)_{k\alpha\beta}^{l\gamma\delta})$ satisfy the normalization condition

$$\hat{F}(a,b,c,1)_{e\alpha\beta}^{f\gamma\delta} = \delta_{\bar{a}}^f \delta_e^{\bar{c}} \delta_\delta^1 \delta_\beta^1 \delta_\alpha^\gamma \tag{4.78}$$

and can be explicitly computed in terms of the $R^\pm$-matrices. □

Dropping the index $\#$, we may rewrite (4.77) as

$$V_\alpha^{nk}(\rho^i)V_\beta^{km}(\rho^j) = \sum_{l,\gamma,\delta} \hat{F}(n,i,j,m)_{k\alpha\beta}^{l\gamma\delta} n(\Gamma_{\rho^i\circ\rho^j,\rho^l}(\gamma)) V_\delta^{nm}(\rho^l). \tag{4.79}$$

Proof. We first note that the definition of fusion on the vacuum sector, Eq. (4.75), is equivalent to the normalization condition (4.78). Let us assume for the moment that ρ^i and ρ^j are spacelike separated. The proof of validity of the general fusion identity, Eq. (4.77), will be reduced to the special case of fusion on the vacuum sector. This requires the introduction of an auxiliary morphism $\rho^{\overline{m}} \in \mathscr{M}_{\overline{m}}^\#$ localized in a cone $\mathscr{C}_{\overline{m}}$ such that if $\mathscr{S}$ is a simple domain containing the localization cones of ρ^i, ρ^j and ρ^l, spacelike to $\mathscr{C}_a^\# + x$, for some $x \in \mathbb{M}^3$, then $\mathscr{S} \not\wr \mathscr{C}_{\overline{m}}$ holds. Clearly two possibilities may occur (see Fig. 4.4):

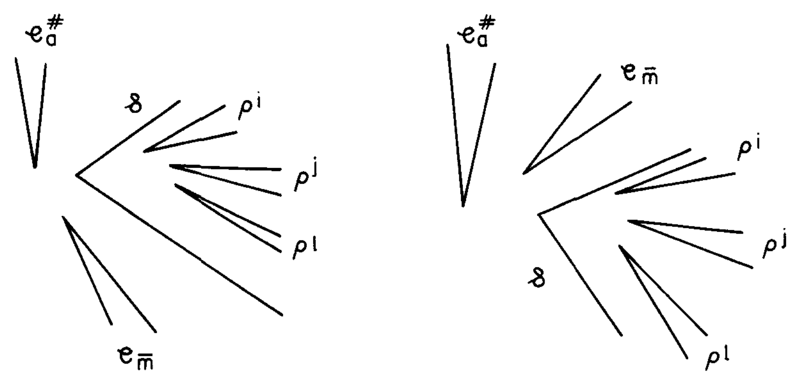


Fig. 4.4

so that either $\mathrm{as}_\# \rho^{\overline{m}} < \mathrm{as}_\# \rho^i$, ρ^j and ρ^l (Fig. 4.4(a)) or $\mathrm{as}_\# \rho^{\overline{m}} > \mathrm{as}_\# \rho^i$, ρ^j and ρ^l hold (Fig. 4.4(b)). We shall study the first case: the construction of the fusion matrices will be shown to be independent of this choice. Let $V_\#^{m1}(\rho^{\overline{m}})$ be the isometry, unique up to a phase, belonging to $\mathscr{V}_m(\rho^{\overline{m}})_1$. We multiply $V_{\alpha\#}^{nk}(\rho^i)V_{\beta\#}^{km}(\rho^j)$ by $V_\#^{m1}(\rho^{\overline{m}})$ and then apply Eq. (4.5) twice:

$$\begin{aligned} V_{\alpha\#}^{nk}(\rho^i)V_{\beta\#}^{km}(\rho^j)V_\#^{m1}(\rho^{\overline{m}}) &= \sum_\gamma R^+(k,j,\overline{m},1)_{m\beta1}^{\overline{j}\gamma1} V_{\alpha\#}^{nk}(\rho^i)V_{\gamma\#}^{k\overline{j}}(\rho^{\overline{m}})V_\#^{\overline{j}1}(\rho^j) \\ &= \sum_{l,\mu,\nu,\gamma} R^+(n,i,\overline{m},\overline{j})_{k\alpha\gamma}^{\overline{l}\mu\nu} R^+(k,j,\overline{m},1)_{m\beta1}^{\overline{j}\gamma1} \\ &\quad \times V_{\mu\#}^{n\overline{l}}(\rho^{\overline{m}})V_{\nu\#}^{\overline{l}\overline{j}}(\rho^i)V_\#^{\overline{j}1}(\rho^j) \end{aligned} \tag{4.80}$$

so that it is now possible to use Eq. (4.75):

$$V_{\alpha\#}^{nk}(\rho^i)V_{\beta\#}^{km}(\rho^j)V_{\#}^{m1}(\rho^{\bar{m}}) = \sum_{l,\mu,\nu,\gamma} R^+(n,i,\bar{m},\bar{j})_{k\alpha\gamma}^{\bar{l}\mu\nu} R^+(k,j,\bar{m},1)_{m\beta 1}^{\bar{j}\gamma 1}$$

$$\times V_{\mu\#}^{n\bar{l}}(\rho^{\bar{m}})\bar{l}\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big)V_{\#}^{\bar{l}1}(\rho^l). \tag{4.81}$$

The intertwining property of $V_\mu^{n\bar{l}}(\rho^{\bar{m}})$ and the localization properties of $\rho^{\bar{m}}$ and of $\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)$ imply that

$$V_{\mu\#}^{n\bar{l}}(\rho^{\bar{m}})\bar{l}\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big) = n\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big)V_{\mu\#}^{n\bar{l}}(\rho^{\bar{m}}). \tag{4.82}$$

Plugging (4.82) into (4.81) and performing an additional permutation, we find

$$V_{\alpha\#}^{nk}(\rho^i)V_{\beta\#}^{km}(\rho^j)V_{\#}^{m1}(\rho^{\bar{m}}) = \sum_{l,\mu,\nu,\gamma} R^+(n,i,\bar{m},\bar{j})_{k\alpha\beta}^{\bar{l}\mu\nu} R^+(k,j,\bar{n},1)_{m\beta 1}^{\bar{j}\gamma 1}$$

$$\times n\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big)V_{\mu\#}^{n\bar{l}}(\rho^{\bar{m}})V_{\#}^{\bar{l}1}(\rho^l)$$

$$= \sum_{l,\mu,\nu,\gamma,\delta} R^-(n,\bar{m},l,1)_{l\mu 1}^{m\delta 1} R^+(n,i,\bar{m},\bar{j})_{k\alpha\gamma}^{\bar{l}\mu\nu} R^+(k,j,\bar{m},1)_{m\beta 1}^{\bar{j}\gamma 1}$$

$$\times n\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big)V_{\delta\#}^{nm}(\rho^l)V_{\#}^{m1}(\rho^{\bar{m}}) \tag{4.83}$$

so that, defining

$$\hat{F}(n,i,j,m)_{k\alpha\beta}^{l\nu\delta} := \sum_{\mu,\gamma} R^-(n,\bar{m},l,1)_{\bar{l}\mu 1}^{m\delta 1} R^+(n,i,\bar{m},\bar{j})_{k\alpha\gamma}^{\bar{l}\mu\nu} R^+(k,j,\bar{m},1)_{m\beta 1}^{\bar{j}\gamma 1}, \tag{4.84}$$

we have that

$$V_{\alpha\#}^{nk}(\rho^i)V_{\beta\#}^{km}(\rho^j)V_{\#}^{m1}(\rho^{\bar{m}}) = \sum_{l,\nu,\delta} \hat{F}(n,i,j,m)_{k\alpha\beta}^{l\nu\delta} n\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big)$$

$$\times V_{\delta\#}^{nm}(\rho^l)V_{\#}^{m1}(\rho^{\bar{m}}). \tag{4.85}$$

Since $\mathscr{V}_m(\rho^{\bar{m}})_1$ is one-dimensional,

$$V_{\alpha\#}^{nk}(\rho^i)V_{\beta\#}^{km}(\rho^j) = \sum_{l,\nu,\delta} \hat{F}(n,i,j,m)_{k\alpha\beta}^{l\nu\delta} n\big(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\nu)\big)V_{\delta\#}^{nm}(\rho^l)$$

holds. This is (4.77). Had we chosen $\mathrm{as}_{\#}\,\rho^{\bar{m}} > \mathrm{as}_{\#}\,\rho^i$, ρ^j, ρ^l instead of $\mathrm{as}_{\#}\,\rho^{\bar{m}} <$ $\mathrm{as}_{\#}\,\rho^i$, ρ^j, ρ^l, then the definition of $\hat{F}(n,i,j,m)_{k\alpha\beta}^{l\nu\delta}$ would have been

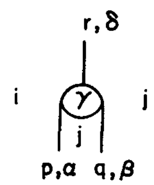
$$\hat{F}(n,i,j,m)_{k\alpha\beta}^{l\nu\delta} = \sum_{\mu,\gamma} R^+(n,\bar{m},l,1)_{\bar{l}\mu 1}^{m\delta 1} R^-(n,i,\bar{m},\bar{j})_{k\alpha\gamma}^{\bar{l}\mu\nu} R^-(k,j,\bar{m},1)_{m\beta 1}^{\bar{j}\gamma 1}. \tag{4.86}$$

It is easy to verify that (4.84) and (4.86) coincide, using the fundamental identity (4.50). Finally, if ρ^i and ρ^j are not spacelike separated, choose $\hat{\rho}^i \in \mathscr{M}_p^{\#}$, unitarily equivalent to ρ^i and spacelike separated from ρ^j. Then

$$V^{nk}_{\alpha\#}(\rho^i)V^{km}_{\beta\#}(\rho^j) = n(\Gamma^{\#}_{\rho^i\rho^i_r}\cdot\Gamma^{\#\,*}_{\hat{\rho}^i\rho^i_r})V^{nk}_{\alpha\#}(\hat{\rho}^i)V^{km}_{\beta\#}(\rho^j) \tag{4.87}$$

holds. We may now apply the previous arguments to the right hand side of (4.87). Using once more the definition (4.74) of $\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\mu)$ one obtains again Eq. (4.77). This completes the proof of Theorem 4.7. ■

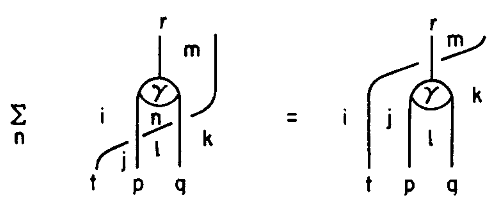
The fusion matrices can be neatly incorporated in the graphical formalism, described in Sec. 4.2 for the $R^{\pm}$-matrices, by introducing the following notation:



$\leftrightarrow \hat{F}(i,p,q,k)^{r\gamma\delta}_{j\alpha\beta}$. (4.88)

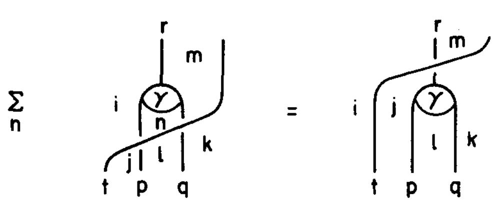
As usual, when reading equations in graphical notation, Greek indices at the tips of identified strings are to be summed over. Sums over Latin indices enclosed in bounded regions will be explicitly indicated, as well as sums over the Greek index γ associated to the vertex of the fusion diagram (4.88).

As in conformal field theory [41, 42, 43, 49] one easily derives the following "polynomial equations" [41, 42, 43, 49] in the present context:

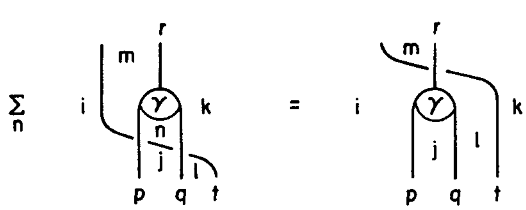


(4.89)

and similarly,



(4.90)



(4.91)

In Eqs. (4.89)–(4.91) we omit Greek indices associated with the tips of strings.

Since the fusion matrices $\hat{F}(m, i, j, n) = \left(\hat{F}(m, i, j, n)_{k\alpha\beta}^{l\gamma\delta}\right)$ provide a unitary transformation between orthonormal bases of $\mathscr{V}_n(\rho^i \circ \rho^j)_m$, they are of course invertible. This means that there exist matrices $\left(\check{F}(n, i, j, m)_{l'\gamma'\delta'}^{k'\alpha'\beta'}\right)$ such that

$$\sum_{k', \alpha', \beta'} \hat{F}(n, i, j, m)_{k'\alpha'\beta'}^{l\gamma\delta} \check{F}(n, i, j, m)_{l'\gamma'\delta'}^{k'\alpha'\beta'} = \delta_{l'}^{l} \delta_{\gamma'}^{\gamma} \delta_{\delta'}^{\delta} \tag{4.92}$$

and

$$\sum_{l, \gamma, \delta} \check{F}(n, i, j, m)_{l\gamma\delta}^{k'\alpha'\beta'} \hat{F}(n, i, j, m)_{k\alpha\beta}^{l\gamma\delta} = \delta_{k}^{k'} \delta_{\alpha}^{\alpha'} \delta_{\beta}^{\beta'} \tag{4.93}$$

hold. The matrices $\check{F}$ arise in the following identities:

$$n\left(\Gamma_{\rho^i \circ \rho^j, \rho^{l'}}(\gamma')\right) V_{\delta'}^{nm}(\rho^{l'}) = \sum_{k', \alpha', \beta'} \check{F}(n, i, j, m)_{l'\gamma'\delta'}^{k'\alpha'\beta'} V_{\alpha'}^{nk'}(\rho^i) V_{\beta'}^{k'm}(\rho^j) \tag{4.94}$$

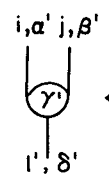
and just as for the $R^{\pm}$-matrices, unitarity means that

$$\check{F}(n, i, j, m)_{l\gamma\delta}^{k\alpha\beta} = \overline{\hat{F}(n, i, j, m)_{k\alpha\beta}^{l\gamma\delta}}. \tag{4.95}$$

The normalization condition (4.78) for the fusion identities on the vacuum sector and Eq. (4.95) imply that $\check{F}(a, b, c, 1)_{e\alpha\beta}^{f\gamma\delta}$ satisfies

$$\check{F}(a, b, c, 1)_{e\alpha\beta}^{f\gamma\delta} = \delta_e^{\bar{a}} \delta_{\bar{c}}^{f} \delta_{\beta}^{1} \delta_{1}^{\delta} \delta_{\alpha}^{\gamma}. \tag{4.96}$$

In analogy with (4.88) we introduce the following graphical notation for the matrices $\check{F}(n, i, j, m)$:



$$\leftrightarrow \check{F}(n, i, j, m)_{l'\gamma'\delta'}^{k'\alpha'\beta'}. \tag{4.97}$$

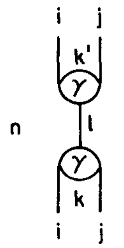
The $\check{F}$-matrices obviously obey polynomial equations analogous to (4.89)–(4.91).

Using the graphical notation (4.88), (4.97), Eqs. (4.92) and (4.93) may be rewritten as

$$\sum_{k'} \text{n}$$

$$\text{m} = \delta_{\gamma'}^{\gamma} \delta_{l}^{l'}\, n \Big|_{l'} m \tag{4.98}$$

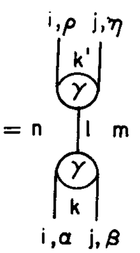
and

$$\sum_{l,\gamma}$$


$$= \delta_k^{k'}\, n \Big| k \Big| m \qquad (4.99)$$
$$\qquad\qquad\quad {\scriptstyle i \;\; j}$$

where we used Eq. (4.43).

Defining the projections $P^{(l,\gamma)}(n,i,j,m) = \left(P^{(l,\gamma)}(n,i,j,m)^{k'\rho\eta}_{k\alpha\beta}\right)$ by

$$P^{(l,\gamma)}(n,i,j,m)^{k'\rho\eta}_{k\alpha\beta} =$$


$$= \sum_{\delta} \hat{F}(n,i,j,m)^{l\gamma\delta}_{k\alpha\beta}\,\check{F}(n,i,j,m)^{k'\rho\eta}_{l\gamma\delta} \qquad (4.100)$$

it is easy to check that Eqs. (4.98) and (4.99) imply orthogonality and completeness relations for these projections:

$$P^{(l,\gamma)}(n,i,j,m)P^{(l',\gamma')}(n,i,j,m) = \delta_{ll'}\delta_{\gamma\gamma'}P^{(l,\gamma)}(n,i,j,m) \qquad (4.101)$$

$$\sum_{l,\gamma} P^{(l,\gamma)}(n,i,j,m) = \mathbb{1}|_{\mathbb{C}^{N^m_{ni\times j}}}. \qquad (4.102)$$

From Eqs. (4.79) and (4.94) one also infers that the complex numbers $P^{(l,\gamma)}(n,i,j,m)^{k'\rho\eta}_{k\alpha\beta}$ are the matrix elements of the projections $n\left(\Gamma^{\#}_{\rho^i\circ\rho^j,\rho^l}(\gamma)\,\Gamma^{\#*}_{\rho^i\circ\rho^j,\rho^l}(\gamma)\right)$, $\# = I, II$ acting on the Hilbert space $\mathscr{H}_n$ with respect to the orthonormal basis $V^{nk}_\alpha(\rho^i)V^{km}_\beta(\rho^j)$ of $\mathscr{V}_n(\rho^i\circ\rho^j)_m$, or, equivalently,

$$n\left(\Gamma_{\rho^i\circ\rho^j,\rho^l}(\gamma)\Gamma^*_{\rho^i\circ\rho^j,\rho^l}(\gamma)\right)V^{nk}_\alpha(\rho^i)V^{km}_\beta(\rho^j)$$
$$= \sum_{k',\rho,\eta} P^{(l,\gamma)}(n,i,j,m)^{k'\rho\eta}_{k\alpha\beta}\,V^{nk'}_\rho(\rho^i)V^{k'm}_\eta(\rho^j). \qquad (4.103)$$

These projections are analyzed in detail in Secs. 5 and 6. Here we merely wish to remark that

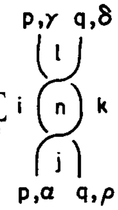
$$P^{(l,\gamma)}(n,i,j,1)^{k'\rho\eta}_{k\alpha\beta} = \delta^l_{\bar{n}}\delta^{\bar{j}}_k\delta^{k'}_{\bar{j}}\delta^1_\beta\delta^1_\eta\delta^\gamma_\alpha\delta^\rho_\gamma\,, \qquad (4.104)$$

an equation which will be useful later on and which follows from (4.78) and (4.96).

Next, we introduce the monodromy matrices

$$M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta} = \sum_{n,\mu,\nu} R^+(i,p,q,k)^{n\mu\nu}_{j\alpha\beta} R^+(i,q,p,k)^{l\gamma\delta}_{n\mu\nu}\,. \tag{4.105}$$

They have the graphical representation

$\sum_n$ 

$$\leftrightarrow M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta}\,. \tag{4.106}$$

Following [24, 43], we prove the following theorem.

Theorem 4.8. *The monodromy and fusion matrices of an algebraic field theory satisfy the equation*

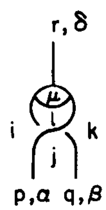
$$\sum_{l,\gamma,\delta} M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta}\hat{F}(i,p,q,k)^{r\mu\nu}_{l\gamma\delta} = e^{2\pi i(s_{\bar{p}}+s_{\bar{q}}-s_{\bar{r}})}\hat{F}(i,p,q,k)^{r\mu\nu}_{j\alpha\beta}\,. \tag{4.107}$$

If $s_j = s_{\bar{j}}$, *Eq.* (4.63) *holds for all sectors* $j \in L$ *we may rewrite* (*Eq.* 4.107) *as*

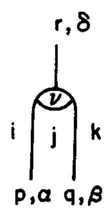
$$\sum_{l,\gamma,\delta} M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta}\hat{F}(i,p,q,k)^{r\mu\nu}_{l\gamma\delta} = e^{2\pi i(s_p+s_q-s_r)}\hat{F}(i,p,q,k)^{r\mu\nu}_{j\alpha\beta}\,. \tag{4.108}$$

□

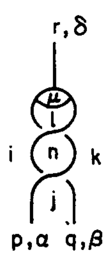
Proof. Let us temporarily assume that

$\sum_l$ 

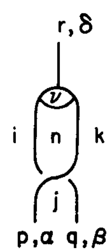
$$= \sum_\nu R^+(\bar{r},p,q,1)^{\bar{p}\mu 1}_{\bar{q}\nu 1} \tag{4.109}$$

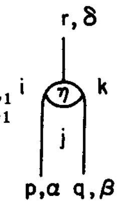


as well as a similar equation for R^- hold. Iterating (4.109), we find that

$\sum_{l,n}$ 

$$= \sum_{n,\nu} R^+(\bar{r},q,p,1)^{\bar{q}\mu 1}_{\bar{p}\nu 1} \tag{4.110}$$



$$= \sum_{\nu,\eta} R^+(\bar{r},q,p,1)^{\bar{q}\mu 1}_{\bar{p}\nu 1} R^+(\bar{r},p,q,1)^{\bar{p}\nu 1}_{\bar{q}\eta 1}$$  . (4.111)

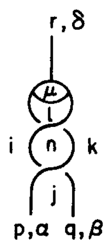
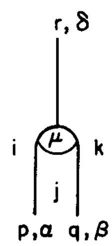
Since, by (4.50)

$$R^+(\bar{r},p,q,1)^{\bar{p}\nu 1}_{\bar{q}\eta 1} = e^{2\pi i(s_{\bar{p}}+s_{\bar{q}}-s_{\bar{r}})} R^-(\bar{r},p,q,1)^{\bar{p}\nu 1}_{\bar{q}\eta 1} \tag{4.112}$$

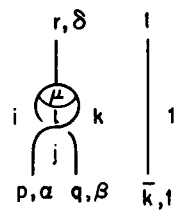
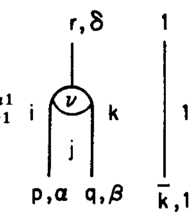
holds, where we have used that $s_1 = 0 \bmod \mathbb{Z}$, and by (4.39)

$$\sum_{\nu} R^+(\bar{r},q,p,1)^{\bar{q}\mu 1}_{\bar{p}\nu 1} R^-(\bar{r},p,q,1)^{\bar{p}\nu 1}_{\bar{q}\eta 1} = \delta^{\mu}_{\eta} \tag{4.113}$$

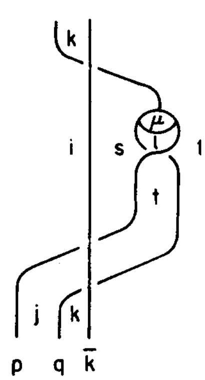
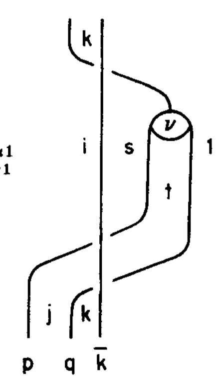
we can rewrite (4.111) as

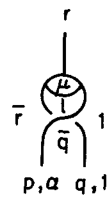
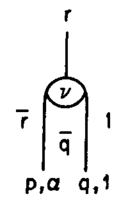
$\sum_{l,n}$  $= e^{2\pi i(s_{\bar{p}}+s_{\bar{q}}-s_{\bar{r}})}$  (4.114)

and this is (4.107). It remains to check identity (4.109). But (4.109) is equivalent to

$\sum_{l}$  $= \sum_{\nu} R^+(\bar{r},p,q,1)^{\bar{p}\mu 1}_{\bar{q}\nu 1}$  (4.115)

and using the polynomial equation (4.90) and equations of Sec. 4.2, we obtain

$\sum_{l,t,s}$  $= \sum_{\nu,s,t} R^+(\bar{r},p,q,1)^{\bar{p}\mu 1}_{\bar{q}\nu 1}$  (4.116) so that it is sufficient to check (4.109) in the case of fusion on the vacuum sector. That is,

$$\sum_{l}$$  $$= \sum_{\nu} R^{+}(\bar{r}, p, q, 1)^{\bar{p}\mu 1}_{\bar{q}\nu 1}$$  (4.117)

The left hand side of this equation reads

$$\sum_{l,\eta,\xi} R^{+}(\bar{r}, p, q, 1)^{l\eta\xi}_{\bar{q}\alpha 1} \hat{F}(\bar{r}, q, p, 1)^{r\mu 1}_{l\eta\xi}$$
$$= \sum_{\eta} R^{+}(\bar{r}, p, q, 1)^{\bar{p}\eta 1}_{\bar{q}\alpha 1} \hat{F}(\bar{r}, q, p, 1)^{r\mu 1}_{\bar{p}\eta 1} \qquad (4.118)$$

where we have used Lemma 4.6. The normalization (4.78) implies that

$$\hat{F}(\bar{r}, q, p, 1)^{r\mu 1}_{\bar{p}\eta 1} = \delta^{\mu}_{\eta} \qquad (4.119)$$

and, plugging (4.119) into (4.118), we find that

$$\sum_{l,\eta,\xi} R^{+}(\bar{r}, p, q, 1)^{l\eta\xi}_{\bar{q}\alpha 1} \hat{F}(\bar{r}, q, p, 1)^{r\mu 1}_{l\eta\xi} = R^{+}(\bar{r}, p, q, 1)^{\bar{p}\mu 1}_{\bar{q}\alpha 1}. \qquad (4.120)$$

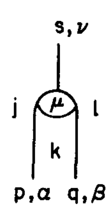
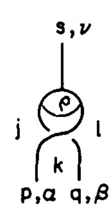
The calculation of the right hand side of (4.117) is just as easy:

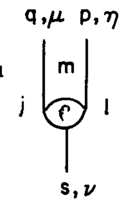
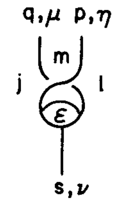
$$\sum_{\gamma} R^{+}(\bar{r}, p, q, 1)^{\bar{p}\mu 1}_{\bar{q}\nu 1} \hat{F}(\bar{r}, p, q, 1)^{r\nu 1}_{\bar{q}\alpha 1} = R^{+}(\bar{r}, p, q, 1)^{\bar{p}\mu 1}_{\bar{q}\alpha 1} \qquad (4.121)$$

using again (4.119). Since (4.120) and (4.121) coincide, this completes the proof of the theorem. ■

In the proof of Theorem 4.8 we have shown that the following lemma holds.

Lemma 4.9.

$$\sum_{\mu} R^{+}(\bar{s}, p, q, 1)^{\bar{p}\rho 1}_{\bar{q}\mu 1}$$  $$=$$  (4.122)

$$\sum_{\rho} R^{+}(\bar{s}, p, q, 1)^{\bar{p}\rho 1}_{\bar{q}\varepsilon 1}$$  $=$  . (4.123)

□

The proof of Eq. (4.123) goes along the same lines as the proof of (4.122).

4.4. *Spin spectrum and statistics of an algebraic field theory, spin addition rules*

Theorem 4.8 allows us to carry over, to the present context, a series of basic results originally proven for conformal field theories; (see [43] and references therein). Two immediate consequences of Eq. (4.108) are that

(i) the fusion matrices $(\hat{F}(i,p,q,k)^{r\mu\nu}_{l\gamma\delta})$ diagonalize the monodromy matrices $M(i,p,q,k)^{l\gamma\delta}_{j\alpha\beta}$.

(ii) The spectrum of $M(i,p,q,k)$ is given by

$$\{e^{2\pi i(s_p+s_q-s_r)} : r \in L, N^r_{pq} \neq 0\}. \tag{4.124}$$

The following theorem has been proven in [43].

Theorem 4.9. *Let us consider an algebraic field theory such that there are only finitely many distinct superselection sectors in L, $|L| < \infty$. Then all the spins s_j, $j \in L$, are rational numbers.* □

In analogy with the terminology used for conformal field theories [44], algebraic field theories with only finitely many distinct superselection sectors, $|L| < \infty$, are called rational theories.

Let us recall from Sec. 4.2 that the statistics occurring in an algebraic field theory is fully characterized by the set of $R^\pm$-matrices defined in Eq. (4.3). If $R^+(i,p,q,k) \neq R^-(i,p,q,k)$ then these $R^\pm$-matrices generate unitary representations of the braid groups. We then speak of *braid statistics*. If $R^+(i,p,q,k) = R^-(i,p,q,k)$ holds for all i, p, q, $k \in L$ then the R-matrices determine unitary representations of the permutation groups, and the statistics is ordinary *permutation statistics*. Theorem 4.8 allows us to characterize the type of statistics encountered in an algebraic field theory (i.e., braid or permutation statistics) in terms of its spin content and vice-versa.

Definition 4.10.

(i) Let $p \in L$ be a superselection sector of an algebraic field theory of spin s_p. We define the *spin parity* of the sector by the equation:

$$\sigma_p = e^{2\pi i s_p}. \tag{4.125}$$

(ii) Let p, q and $r \in L$ be superselection sectors satisfying the fusion rule $N^r_{pq} \neq 0$. We say that spin parity is *conserved* for the fusion of p and q into r if

$$\sigma_p \cdot \sigma_q = \sigma_r \tag{4.126}$$

holds.

If (4.126) holds for *arbitrary* p, q and r in L, with $N_{pq}^r \neq 0$, we say that spin parity is conserved. □

Theorem 4.11. *For an algebraic field theory, the following two properties are equivalent*:

(i) *The statistics of the theory is ordinary permutation statistics.*

(ii) *Spin parity is conserved under fusion.*

Furthermore, both properties imply that

$$s_p \in \frac{1}{2}\mathbb{Z}, \tag{4.127}$$

where s_p is the spin of the superselection sector $p \in L$. □

Proof. Let us assume that $R^+(i,p,q,k) = R^-(i,p,q,k)$ holds for all $i,p,q,k \in L$. Then one concludes from Eq. (4.39) that the monodromy matrices are trivial,

$$M(i,p,q,k)_{j\alpha\beta}^{l\gamma\delta} = \delta_j^l \delta_\alpha^\gamma \delta_\beta^\delta, \tag{4.128}$$

and hence all their eigenvalues are equal to one,

$$e^{2\pi i(s_p + s_q - s_r)} = 1 \tag{4.129}$$

for all p, q and r in L with $N_{pq}^r \neq 0$ (see Eq. (4.124)). This last equation is equivalent to conservation of spin parity. Conversely, conservation of spin parity implies that all eigenvalues of the monodromy matrices are trivial, i.e., Eq. (4.128) holds. By Eq. (4.105) we conclude that

$$R^+(i,p,q,k) = R^-(i,p,q,k). \tag{4.130}$$

To show that (i) and (ii) imply that all spins of the theory are half-integral, let us consider Eq. (4.129) for the case that $q = \bar{p}$ and $r = 1$. By definition of the conjugate charge we have that $N_{p\bar{p}}^1 \neq 0$. Moreover $s_p = s_{\bar{p}}$.

Hence, we conclude from (4.129) that

$$e^{4\pi i s_p} = 1 \tag{4.131}$$

or

$$s_p \in \frac{1}{2}(\mathrm{mod}\,\mathbb{Z}). \tag{4.132}$$

This completes the proof of the theorem. ■

In three-dimensional theories with charges localizable in double cones, one easily sees [17, 22] that ordinary permutation statistics occurs:

$$R^+(i,p,q,k) = R^-(i,p,q,k) \tag{4.133}$$

for all i, p, q, k in L. Hence, by Theorem 4.11, all sectors of such theories have integral or half-integral spin and spin parity is conserved under fusion.

Next, we analyze theories with *abelian* braid group statistics, i.e., theories for which the statistics matrices generate a one-dimensional representation of the braid groups, for some sector $p \in L$. It has interesting applications in quantum field theory and in condensed matter physics [58]. Our purpose is to derive a spin addition rule for abelian sectors which has been conjectured in [45] on the basis of an analysis of anyon models.

As shown in [23] by a straightforward generalization of an argument given in [17, 18], abelian braid group statistics for the sector $p \in L$ implies that all morphisms $\rho^p \in \mathcal{M}_p^\#$ are *-automorphisms of the algebra $\mathcal{B}^\#$, $\# = I, II$. In that case, $\bar{\rho} = \rho^{-1}$ and hence $\bar{p} \times p = 1$ holds. Every power $p^{\times n} = p \times \cdots \times p$ (n times), $n \in \mathbb{Z}$, of the representation p is irreducible and belongs to the list L. The subset $\tilde{L} = \{p^{\times n}; n \in \mathbb{Z}\}$ of L is invariant under composition and charge conjugation, its fusion rules are described by

$$N_{p^{\times m}p^{\times n}}^{p^{\times l}} = \begin{cases} 1 & \text{if } l = n + m, \\ 0 & \text{otherwise}. \end{cases} \tag{4.134}$$

All intertwiner spaces for representations in $\tilde{L}$ are one-dimensional and Schur's lemma implies that

$$R^\pm(l,m,n,k)_j^i = \begin{cases} e^{2\pi i\theta_{m,n}} & \text{if } N_{lm}^j N_{jn}^n \neq 0 \text{ and } N_{ln}^i N_{im}^k \neq 0, \\ 0 & \text{otherwise}, \end{cases} \tag{4.135}$$

for l, m, n, k, i, j in $\tilde{L}$ and $\theta_{k,1} = \theta_{1,k} = 1$. It then follows from Eq. (4.91) (with $q = \bar{p}, t = p, r = 1, k = 1, l = \bar{p}, j = 1, i = \bar{p}$ and $m = 1$) that

$$R^+(1,\bar{p},p,1)_{\bar{p}}^p R^+(\bar{p},p,p,p)_1^1 = 1 \tag{4.136}$$

or

$$e^{2\pi i\theta_{\bar{p},p}} \cdot e^{2\pi i\theta_{p,p}} = 1. \tag{4.137}$$

By (4.135),

$$R^+(\bar{p} \times \bar{p},p,p,1)_{\bar{p}}^{\bar{p}} = e^{2\pi i\theta_{p,p}}. \tag{4.138}$$

Furthermore, (4.106) and (4.108) show that

$$(R^+(\bar{p} \times \bar{p},p,p,1)_{\bar{p}}^{\bar{p}})^2 = e^{2\pi i(2s_p - s_{p\times p})}. \tag{4.139}$$

Combining (4.136)–(4.139) we find

$$e^{2\pi i s_{p\times p}} = e^{2\pi i(2s_p + 2\theta_{\bar{p},p})} \tag{4.140}$$

and by (4.62)

$$e^{2\pi i s_{p\times p}} = e^{2\pi i(4s_p)} \tag{4.141}$$

or

$$s_{p\times p} = 4s_p(\mathrm{mod}\,\mathbb{Z}). \tag{4.142}$$

Iterating these arguments, one finds that

$$s_{p^{\times n}} = n^2 s_p\,(\mathrm{mod}\,\mathbb{Z}). \tag{4.143}$$

This is the desired spin addition rule for abelian sectors. Similarly,

$$\theta_{p^{\times n},\bar{p}^{\times n}} = n^2\theta_{p,\bar{p}}.$$

In the non-abelian case, analogous spin addition rules can be proven by using (4.62) and the polynomial equation (4.91), provided the fusion matrices $\{\hat{F}(i,m,n,k)_j^r\}$ can be calculated without using Eq. (4.84) (see [23]). This is the case if, for example, the $R^{\pm}$ and $\hat{F}$ matrices can be derived from the representation theory of some (quasi-)quantum group via the vertex-sos transformation [43].

5. The Statistics Parameter of a Superselection Sector $p \in L$

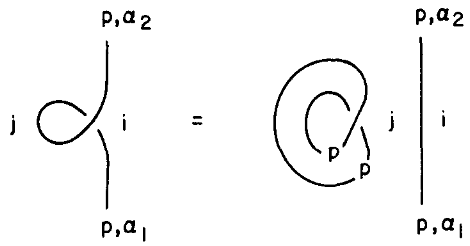
In this section, we define a numerical invariant for $p \in L$, called the statistics parameter of a superselection sector p. The statistics parameter, denoted by λ_p, plays a central role for theories with permutation statistics. Doplicher, Haag and Roberts [17] showed that λ_p is always of the form

$$\lambda_p = \pm\frac{1}{d(p)}, \qquad d(p) \in \{1,2,3,\dots\} \tag{5.1}$$

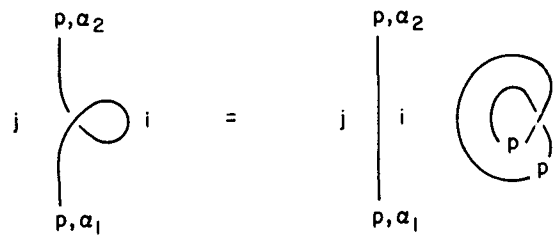
and that representations of the permutation groups arising in this case (see Sec. 6) are classified up to unitary quasi-equivalence by λ_p. The importance of the invariant λ_p is illustrated by a general result of Longo [30] relating it to the index of the inclusion $\rho(\mathscr{A}(\mathscr{C})^{-w}) \subseteq \mathscr{A}(\mathscr{C})^{-w}$, and a substantial portion of the next two sections is devoted to analyzing its role in the present context.

We start by proving a basic lemma.

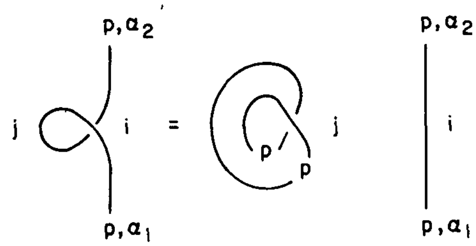
Lemma 5.1. *The following graphical equations hold.*



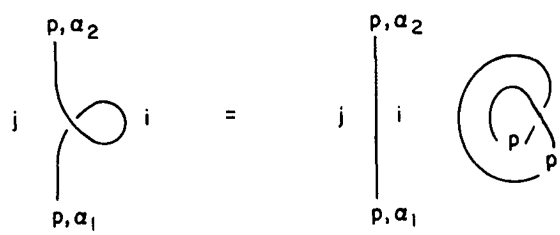
(5.2)



(5.3)



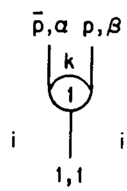
(5.4)



(5.5)

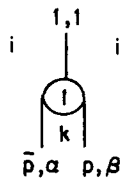
□

In this graphical Eqs. (5.2)–(5.5) we use several new conventions: the following fusion matrices



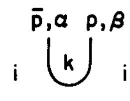
$$\leftrightarrow \check{F}(i,\bar{p},p,i)^{k\alpha\beta}_{111}\,, \tag{5.6}$$

and

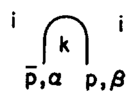


$$\leftrightarrow \hat{F}(i,\bar{p},p,i)^{111}_{k\alpha\beta} \tag{5.7}$$

have been replaced by



$$\leftrightarrow \check{F}(i,\bar{p},p,i)^{k\alpha\beta}_{111}, \tag{5.8}$$

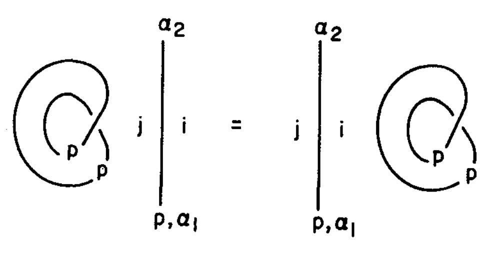


$$\leftrightarrow \hat{F}(i,\bar{p},p,i)^{111}_{k\alpha\beta}. \tag{5.9}$$

This is justified by the fact that the position of a string assigned to the vacuum representation 1 is irrelevant in a graphical equation if one makes the natural choice $V^{pp}(1) = \mathbb{1}|_{\mathscr{H}_p}$, for all $p \in L$. Sums over Latin indices of bounded regions have also been suppressed. In terms of braiding and fusion matrices, Eq. (5.2), for example, reads

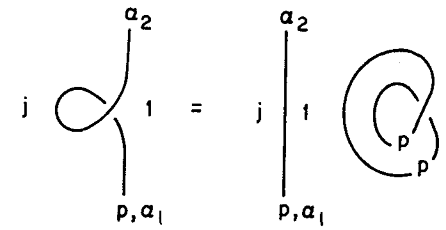
$$\begin{aligned}
&\sum_{s_1,\beta,\gamma_1,\gamma_2} \check{F}(j,\bar{p},p,j)^{s\beta\gamma_1}_{111} R^{+}(s,p,p,i)^{j\gamma_2\alpha_2}_{j\gamma_1\alpha_1} \hat{F}(j,\bar{p},p,j)^{111}_{s\beta\gamma_2} \\
&= \sum_{\substack{s_1,\beta_1,\gamma_1,\\ s_2,\beta_2,\gamma_2,\\ \gamma_3,\gamma_4}} \check{F}(j,\bar{p},p,j)^{s_1\beta_1\gamma_1}_{111} \check{F}(s_1,\bar{p},p,s_1)^{s_2\beta_2\gamma_2}_{111} R^{+}(s_2,p,p,j)^{s_1\gamma_3\gamma_4}_{s_1\gamma_2\gamma_1} \\
&\quad \times \hat{F}(s_1,\bar{p},p_1,s_1)^{111}_{s_2\beta_2\gamma_3} \hat{F}(j,\bar{p},p,j)^{111}_{s_1\beta_1\gamma_4} \delta^{\alpha_2}_{\alpha_1}.
\end{aligned} \tag{5.10}$$

Proof. We will only prove Eq. (5.2), since the proof of the other equations is analogous. The polynomial and Yang-Baxter equations imply that



(5.11)

holds. We prove Eq. (5.2) in the special case $i = 1$. By (5.11), it is sufficient to check the following equality:



(5.12)

The fusion rules imply that $j = \bar{p}$, $\alpha_1 = \alpha_2 = 1$ and the left hand side of (5.12) is

$$\sum_{s,\beta,\gamma_1,\gamma_2} \check{F}(\bar{p},\bar{p},p,\bar{p})^{s\beta\gamma_1}_{111} R^+(s,p,p,1)^{\bar{p}\gamma_2 1}_{\bar{p}\gamma_1 1} \hat{F}(\bar{p},\bar{p},p,\bar{p})^{111}_{s\beta\gamma_2}. \tag{5.13}$$

For the right hand side, we obtain

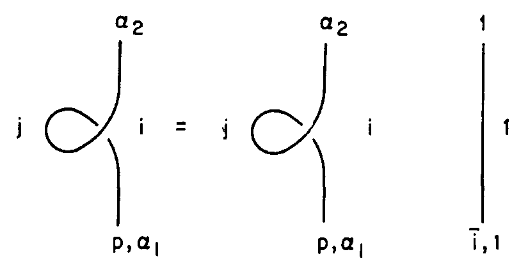
$$\sum_{\substack{s_1,\beta_1,\gamma_1,\\ s_2,\beta_2,\gamma_2,\\ \gamma_3,\gamma_4}} \check{F}(1,\bar{p},p,1)^{s_1\beta_1\gamma_1}_{111} \check{F}(s_1,\bar{p},p,s_1)^{s_2\beta_2\gamma_2}_{111} R^+(s_2,p,p,1)^{s_1\gamma_3\gamma_4}_{s_1\gamma_2\gamma_1}$$
$$\times \hat{F}(s_1,\bar{p},p,s_1)^{111}_{s_2\beta_2\gamma_3} \hat{F}(1,\bar{p},p,1)^{111}_{s_1\beta_1\gamma_4} \cdot 1. \tag{5.14}$$

But $\check{F}(1,\bar{p},p,1)^{s_1\beta_1\gamma_1}_{111} = \delta^{s_1}_{\bar{p}}\delta^{\beta_1}_{1}\delta^{\gamma_1}_{1}$ and $\hat{F}(1,\bar{p},p,1)^{111}_{s_1\beta_1\gamma_4} = \delta^{\bar{p}}_{s_1}\delta^{1}_{\beta_1}\delta^{1}_{\gamma_4}$ by Eqs. (4.78) and (4.96), so that (5.14) reduces to

$$\sum_{s_2,\beta_2,\gamma_2,\gamma_3} \check{F}(\bar{p},\bar{p},p,\bar{p})^{s_2\beta_2\gamma_2}_{111} R^+(s_2,p,p,1)^{\bar{p}\gamma_3 1}_{\bar{p}\gamma_2 1} \hat{F}(\bar{p},\bar{p},p,\bar{p})^{111}_{s_2\beta_2\gamma_3} \tag{5.15}$$

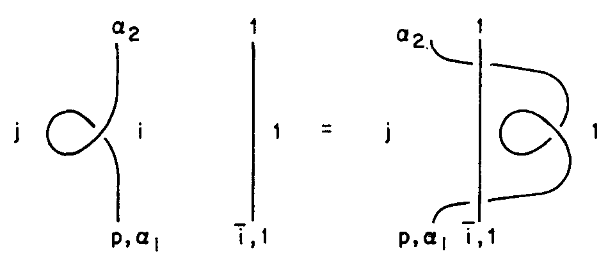
which is (5.13).

To prove the general case, we note that



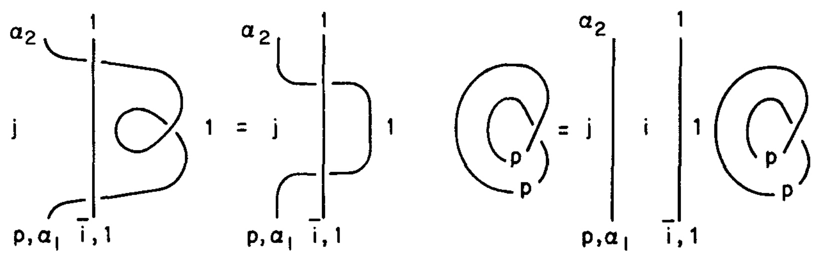
(5.16)

Using the Yang-Baxter and polynomial equations, we obtain



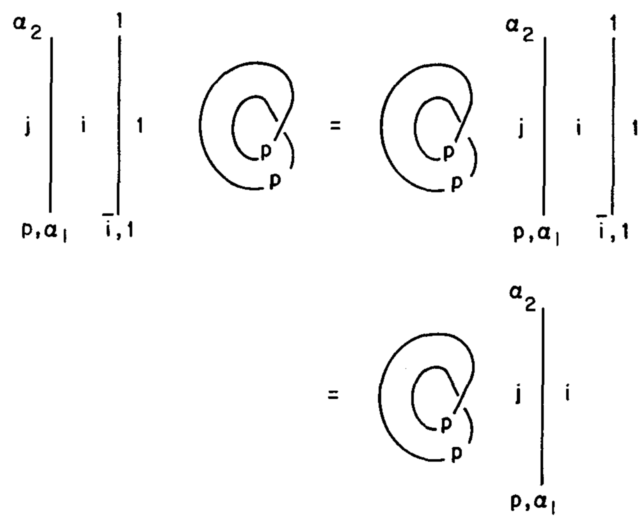
(5.17)

Applying the result (5.12),



(5.18)

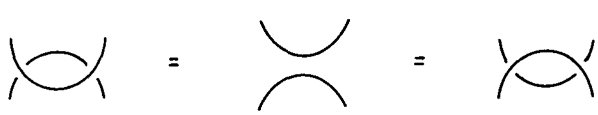
where the second equality follows again from the Yang-Baxter equation. Finally, applying a last time the Yang-Baxter and polynomial equations, we obtain



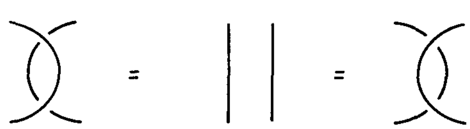
(5.19)

which completes the proof. ■

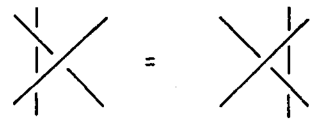
It should be clear that the new notation introduced in the formulation of Lemma 5.1 considerably simplifies the proof of equations in the graphical formalism: the Yang-Baxter and polynomial equations imply



(5.20)



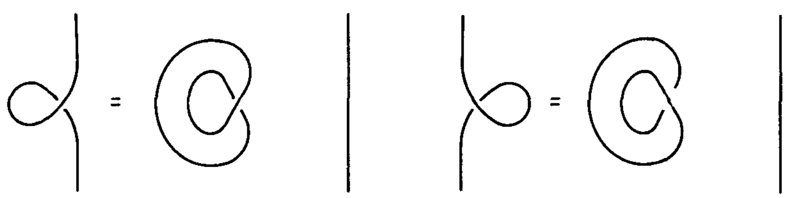
(5.21)



(5.22)

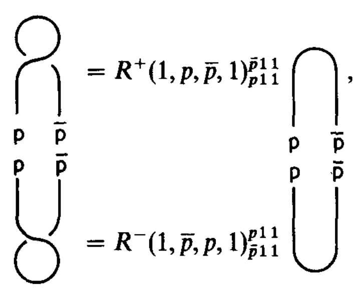
(5.23)

Equations (5.20) and (5.21) mean that our graphical equations are invariant under Reidemeister moves of type II, just as the usual diagrams for knots and links [54]. Equations of the type (5.22) and (5.23) correspond to invariance under Reidemeister moves of type III. The transformation properties under Reidemeister moves of type I are described by Lemmas 4.9 and 5.1.



(5.24)

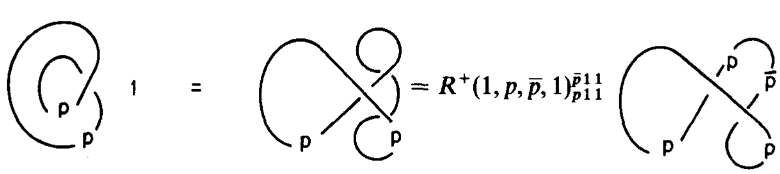
and

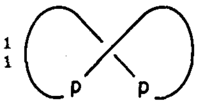


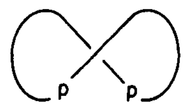
(5.25)

These facts were first noticed in the context of conformal field theories [55] and were employed to construct invariants for knots and links generalizing Jones' invariant [56]. They will also lead to link invariants in the present context [24].

We now apply Eqs. (5.20)–(5.25) to compute



$$= R^+(1,p,\bar{p},1)^{\bar{p}11}_{p11} R^-(1,\bar{p},p,1)^{p11}_{\bar{p}11}$$ 

$$=$$ 

$$= R^+(\bar{p},p,p,p)^{111}_{111}. \tag{5.26}$$

Definition 5.2. *The complex number $R^+(\bar{p},p,p,p)^{111}_{111}$ is called the statistics parameter of the sector p and we write*

$$\lambda_p = R^+(\bar{p},p,p,p)^{111}_{111}. \tag{5.27}$$

□

From Eq. (4.40) it follows that

$$\bar{\lambda}_p = R^-(\bar{p},p,p,p)^{111}_{111}. \tag{5.28}$$

Remark 5.3. $R^+(\bar{p},p,p,p)^{111}_{111}$ is a matrix element of the statistics operator $\bar{p}(\varepsilon^+_{\rho^p,\rho^p})$, because, by Eq. (4.21),

$$R^+(\bar{p},p,p,p)^{111}_{111} = \langle V^{\bar{p}1}(\rho^p)V^{1p}(\rho^p); \bar{p}(\varepsilon^+_{\rho^p,\rho^p})V^{\bar{p}1}(\rho^p)V^{1p}(\rho^p)\rangle. \tag{5.29}$$

Applying a unitary "gauge" transformation of the type (4.34) in the one-dimensional spaces $\mathscr{V}_{\bar{p}}(\rho^p)_1$, $\mathscr{V}_1(\rho^p)_p$,

$$\begin{aligned} V^{\bar{p}1}(\rho^p) &\to e^{i\theta_1}V^{\bar{p}1}(\rho^p), \\ V^{1p}(\rho^p) &\to e^{i\theta_2}V^{1p}(\rho^p) \end{aligned} \tag{5.30}$$

one checks, using Eq. (4.35), that the matrix element $R^+(\bar{p},p,p,p)^{111}_{111}$ is invariant under this gauge transformation. □

Lemma 5.4.

$$R^+(\bar{p},p,p,p)^{111}_{111} = \hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}R^-(1,p,\bar{p},1)^{\bar{p}11}_{p11}. \tag{5.31}$$

□

Proof. Equation (5.31) is an immediate consequence of the definition of fusion matrices, Eq. (4.84):

$$\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} = \sum_{\mu,\gamma} R^-(\bar{p},p,1,1)^{\bar{p}\delta 1}_{1\mu 1} R^+(\bar{p},p,p,p)^{1\mu 1}_{11\gamma} R^+(1,\bar{p},p,1)^{p\gamma 1}_{\bar{p}11} \tag{5.32}$$

since

$$R^+(1,\overline{p},p,1)^{p\gamma 1}_{\overline{p}11} = R^+(1,\overline{p},p,1)^{p11}_{\overline{p}11}\delta^\gamma_1 \tag{5.33}$$

holds and moreover the equations

$$V^{\overline{p}1}(\rho^p)V^{11}(1) = R^-(\overline{p},p,1,1)^{111}_{111}V^{\overline{p}\overline{p}}(1)V^{\overline{p}1}(\rho^p), \tag{5.34}$$

$$V^{11}(1) = \mathbb{1}|_{\mathscr{H}_1}, \qquad V^{\overline{p}\overline{p}}(1) = \mathbb{1}|_{\mathscr{H}_{\overline{p}}} \tag{5.35}$$

imply that

$$R^-(\overline{p},p,1,1)^{\overline{p}\delta 1}_{1\mu 1} = \delta^1_\mu\delta^\delta_1, \tag{5.36}$$

we obtain

$$\hat{F}(\overline{p},p,\overline{p},\overline{p})^{111}_{111} = R^+(\overline{p},p,p,p)^{111}_{111}R^+(1,\overline{p},p,1)^{p11}_{\overline{p}11} \tag{5.37}$$

which is (5.31). ■

By definition of the conjugate sector, the representation $\overline{p}\times p\times\overline{p}$ contains the representation $\overline{p}\in L$ at least once as a subrepresentation, so that the constant $\hat{F}(\overline{p},p,\overline{p},\overline{p})^{111}_{111}$ is non-vanishing.

Lemma 5.5.

$$\overline{\rho^p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})\Gamma_{\overline{\rho^p}\circ\rho^p,1} = \hat{F}(\overline{p},p,\overline{p},\overline{p})^{111}_{111}\cdot\mathbb{1}|_{\mathscr{H}_1}, \tag{5.38}$$

$$\rho^p(\Gamma^*_{\overline{\rho^p}\circ\rho^p,1})\Gamma_{\rho^p\circ\overline{\rho^p},1} = \hat{F}(p,\overline{p},p,p)^{111}_{111}\cdot\mathbb{1}|_{\mathscr{H}_1}. \tag{5.39}$$

□

Proof. We prove only Eq. (5.38). Equation (5.39) follows by exchanging p and $\overline{p}$. The irreducibility of the representation $\overline{\rho^p}$ of $\mathscr{A}$ on the vacuum sector and the fact that

$$\overline{\rho^p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})\Gamma_{\overline{\rho^p}\circ\rho^p,1} \in \overline{\rho^p}(\mathscr{A})' \tag{5.40}$$

imply that

$$\overline{\rho^p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})\Gamma_{\overline{\rho^p}\circ\rho^p,1} = \mu\cdot\mathbb{1}|_{\mathscr{H}_1} \tag{5.41}$$

for some $\mu\in\mathbb{C}$. To evaluate this constant, we consider the action of $\overline{\rho^p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})\,\Gamma_{\overline{\rho^p}\circ\rho^p,1}$ on the one-dimensional intertwiner space $\mathscr{V}_1(\overline{\rho^p})_{\overline{p}}$:

$$\begin{aligned}\overline{\rho^p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})\Gamma_{\overline{\rho^p}\circ\rho^p,1}V^{1\overline{p}}(\overline{\rho^p}) &= \overline{\rho^p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})V^{1\overline{p}}(\overline{\rho^p})V^{\overline{p}1}(\rho^p)V^{1\overline{p}}(\overline{\rho^p})\\ &= V^{1\overline{p}}(\overline{\rho^p})\overline{p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})V^{\overline{p}1}(\rho^p)V^{1\overline{p}}(\overline{\rho^p}),\end{aligned} \tag{5.42}$$

where the first equality follows from Eq. (4.75) and the second from the intertwining properties of $V^{1\bar{p}}(\overline{\rho^p})$, since

$$V^{\bar{p}1}(\rho^p)V^{1\bar{p}}(\overline{\rho^p}) = \sum_{k,\alpha,\beta} \hat{F}(\bar{p},p,\bar{p},\bar{p})^{k\alpha\beta}_{111}\bar{p}(\Gamma_{\rho^p\circ\overline{\rho^p},p^l}(\alpha))V^{\bar{p}\bar{p}}_{\beta}(\rho^l) \tag{5.43}$$

we obtain

$$\bar{p}(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})V^{\bar{p}1}(\rho^p)V^{1\bar{p}}(\overline{\rho^p}) = \hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}\,. \tag{5.44}$$

Inserting (5.44) into (5.42) we see that

$$\mu = \hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}\,. \tag{5.45}$$

■

Our next goal is to show that the constants $\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$ and $\hat{F}(p,\bar{p},p,p)^{111}_{111}$ occuring in Eqs. (5.38) and (5.39) may be chosen to be simultaneously positive by an adequate normalization of the orthonormal bases in the intertwiner spaces $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\overline{\rho^p})_{\bar{p}}$ and $\mathscr{V}_{\bar{p}}(\rho^p)_1$. This will identify $\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$ with the absolute value of the statistics parameter and shown that $|\lambda_p| = |\lambda_{\bar{p}}|$ holds. The proof of those results require some preliminary computations. The next lemma determines the transformation properties of $\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$ under a unitary "gauge" transformation of the type (4.34).

Lemma 5.6. *Let $V^{p1}(\overline{\rho^p})$, $V^{1p}(\rho^p)$, $V^{\bar{p}1}(\rho^p)$ and $V^{1\bar{p}}(\overline{\rho^p})$ be (ortho-)normal vectors in the one-dimensional spaces $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_{\bar{p}}(\rho^p)_1$ and $\mathscr{V}_1(\overline{\rho^p})_{\bar{p}}$. If we perform the unitary "gauge" transformations*:

$$\begin{aligned}
&V^{p1}(\overline{\rho^p}) \to e^{i\varphi_1}V^{p1}(\overline{\rho^p}) =: \tilde{V}^{p1}(\overline{\rho^p})\\
&V^{1p}(\rho^p) \to e^{i\varphi_2}V^{1p}(\rho^p) =: \tilde{V}^{1p}(\rho^p)\\
&V^{\bar{p}1}(\rho^p) \to e^{i\varphi_3}V^{\bar{p}1}(\rho^p) =: \tilde{V}^{\bar{p}1}(\rho^p)\\
&V^{1\bar{p}}(\overline{\rho^p}) \to e^{i\varphi_4}V^{1\bar{p}}(\rho^p) =: V^{1\bar{p}}(\rho^p)
\end{aligned} \tag{5.46}$$

in the intertwiner spaces $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_{\bar{p}}(\rho^p)_1$ and $\mathscr{V}_1(\overline{\rho^p})_{\bar{p}}$ then the new fusion matrix elements $\hat{\hat{F}}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$, $\hat{\hat{F}}(p,\bar{p},p,p)^{111}_{111}$ computed with respect to the new (ortho-)normal bases $\tilde{V}^{p1}(\overline{\rho^p})$, $\tilde{V}^{1p}(\rho^p)$, $\tilde{V}^{\bar{p}1}(\rho^p)$ and $\tilde{V}^{1\bar{p}}(\overline{\rho^p})$ are related to the old matrix elements $\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$, $\hat{F}(p,\bar{p},p,p)^{111}_{111}$ by

$$\hat{\hat{F}}(p,\bar{p},p,p)^{111}_{111} = e^{i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\hat{F}(p,\bar{p},p,p)^{111}_{111} \tag{5.47}$$

$$\hat{\hat{F}}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} = e^{-i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}\,. \tag{5.48}$$

□

Proof. By Lemma 5.4 we know that

$$\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} = R^{+}(\bar{p},p,p,p)^{111}_{111}R^{+}(1,\bar{p},p,1)^{p11}_{\bar{p}11} \tag{5.49}$$

and

$$\hat{F}(p,\bar{p},p,p)^{111}_{111} = R^{+}(p,\bar{p},\bar{p},\bar{p})^{111}_{111}R^{+}(1,p,\bar{p},1)^{\bar{p}11}_{p11}\,. \tag{5.50}$$

Since the matrix elements $R^{+}(\bar{p},p,p,p)^{111}_{111}$ and $R^{+}(p,\bar{p},\bar{p},\bar{p})^{111}_{111}$ are invariant under the "gauge" transformations (5.46), it suffices to prove that the new matrix elements $\tilde{R}(1,\bar{p},p,1)^{p11}_{\bar{p}11}$ and $\tilde{R}^{+}(1,p,\bar{p},1)^{\bar{p}11}_{p11}$ are related to the old ones by

$$\begin{aligned}\tilde{R}^{+}(1,\bar{p},p,1)^{p11}_{\bar{p}11} &= e^{-i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}R^{+}(1,\bar{p},p,1)^{p11}_{\bar{p}11}\,,\\ \tilde{R}^{+}(1,p,\bar{p},1)^{\bar{p}11}_{p11} &= e^{i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}R^{+}(1,p,\bar{p},1)^{\bar{p}11}_{p11}\,.\end{aligned} \tag{5.51}$$

But, by Eq. (4.21),

$$\begin{aligned}R^{+}(1,\bar{p},p,1)^{p11}_{\bar{p}11} &= \langle V^{1p}(\rho^{p})V^{p1}(\overline{\rho^{p}});\varepsilon^{+}_{\overline{\rho^{p}},\rho^{p}}V^{1\bar{p}}(\overline{\rho^{p}})V^{\bar{p}1}(\rho^{p})\rangle\\ R^{+}(1,p,\bar{p},1)^{\bar{p}11}_{p11} &= \langle V^{1\bar{p}}(\overline{\rho^{p}})V^{\bar{p}1}(\rho^{p});\varepsilon^{+}_{\rho^{p},\overline{\rho^{p}}}V^{1p}(\rho^{p})V^{p1}(\overline{\rho^{p}})\rangle\,.\end{aligned} \tag{5.52}$$

The equations (5.51) follow by comparing (5.52) with the analogous equations for $\tilde{R}^{+}(1,p,\bar{p},1)^{\bar{p}11}_{p11}$, $\tilde{R}^{+}(1,\bar{p},p,1)^{p11}_{\bar{p}11}$ using Eq. (5.46). ■

The next lemma will be important.

Lemma 5.7.

$$\hat{F}(p,\bar{p},p,p)^{111}_{111} = \overline{\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}}\,. \tag{5.53}$$

□

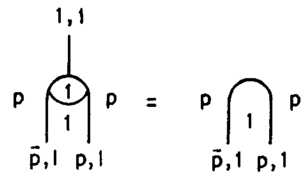
Proof. We already proved that

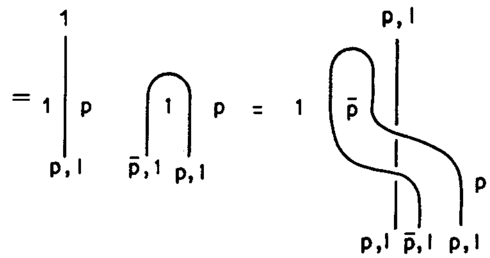
$$\begin{aligned}\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} &= R^{+}(\bar{p},p,p,p)^{111}_{111}R^{+}(1,\bar{p},p,1)^{p11}_{\bar{p}11}\\ &= \overline{R^{-}(\bar{p},p,p,p)^{111}_{111}}\;\overline{R^{-}(1,p,\bar{p},1)^{\bar{p}11}_{p11}}\end{aligned} \tag{5.54}$$

where (5.54) follows by use of (4.40). If suffices to show that

$$\hat{F}(p,\bar{p},p,p)^{111}_{111} = R^{-}(\bar{p},p,p,p)^{111}_{111}R^{-}(1,p,\bar{p},1)^{\bar{p}11}_{p11}\,.$$

But, using our graphical notation,

$\hat{F}(p,\bar{p},p,p)^{111}_{111} =$ 



$$= R^{-}(1,p,\bar{p},1)^{\bar{p}11}_{p11} R^{-}(\bar{p},p,p,p)^{111}_{111},$$

since $\hat{F}(1,\bar{p},p,1)^{111}_{\bar{p}11} = 1$ by normalization of the fusion matrices on the vacuum sector. This completes the proof of Lemma 5.7. ■

As a corollary of Lemmas 5.6 and 5.7 we have the following theorem.

Theorem 5.8. *There exist bases in* $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_{\bar{p}}(\rho^p)_1$ *and* $\mathscr{V}_1(\overline{\rho^p})_{\bar{p}}$ *such that the corresponding fusion matrix elements* $\hat{F}(p,\bar{p},p,p)^{111}_{111}$ and $\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$ *are positive.* □

Proof. Let $V^{p1}(\overline{\rho^p})$, $V^{1p}(\rho^p)$, $V^{\bar{p}1}(\rho^p)$ and $V^{1\bar{p}}(\overline{\rho^p})$ be arbitrary bases in $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_{\bar{p}}(\rho^p)_1$ and $\mathscr{V}_1(\rho^p)_{\bar{p}}$; let $\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111}$, $\hat{F}(p,\bar{p},p,p)^{111}_{111}$ be the corresponding fusion matrix elements. By an appropriate "gauge" transformation (5.46) we can achieve that

$$0 < \hat{\hat{F}}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} \tag{5.55}$$

holds. It then follows from (5.48), (5.53) and (5.47) that

$$\begin{aligned} 0 < \hat{\hat{F}}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} &= e^{-i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\hat{F}(\bar{p},p,\bar{p},\bar{p})^{111}_{111} \\ &= e^{-i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\overline{\hat{F}(p,\bar{p},p,p)^{111}_{111}} \\ &= \overline{e^{i(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\hat{F}(p,\bar{p},p,p)^{111}_{111}} \\ &= \overline{\hat{\hat{F}}(p,\bar{p},p,p)^{111}_{111}} \end{aligned}$$

and hence

$$0 < \overline{\hat{\hat{F}}(p,\bar{p},p,p)^{111}_{111}} = \hat{\hat{F}}(p,\bar{p},p,p)^{111}_{111}. \tag{5.56}$$

This completes the proof of the theorem. ■

Corollary 5.9. *For the bases of* $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_{\bar{p}}(\rho^p)_1$ *and* $\mathscr{V}_1(\overline{\rho^p})_{\bar{p}}$ *constructed in Theorem* 5.8,

(i)
$$|\lambda_p| = \hat{F}(\bar{p}, p, \bar{p}, \bar{p})^{111}_{111} \tag{5.57}$$

Furthermore,

(ii)
$$|\lambda_p| = |\lambda_{\bar{p}}|. \tag{5.58}$$

□

Proof. By Eq. (5.31),

$$\begin{aligned}\lambda_p &= \hat{F}(\bar{p}, p, \bar{p}, \bar{p})^{111}_{111} R^-(1, p, \bar{p}, 1)^{\bar{p}11}_{p11} \\ &= \hat{F}(\bar{p}, p, \bar{p}, \bar{p})^{111}_{111} e^{-2\pi i \theta_{\bar{p},p}},\end{aligned}$$

using the notation introduced in (4.56). Taking absolute values on both sides proves (i); (ii) follows from Eq. (5.56). ■

Next, we begin the proof of Lemma 4.6 (iii). The first step is to define an antilinear operator between the intertwiner spaces $\mathscr{V}_j(\rho^p)_k$ and $\mathscr{V}_k(\overline{\rho^p})_j$, for k, j and $p \in L$. From now on, we always use basis vectors in the intertwiner spaces $\mathscr{V}_p(\overline{\rho^p})_1$, $\mathscr{V}_1(\rho^p)_p$, $\mathscr{V}_{\bar{p}}(\rho^p)_1$ and $\mathscr{V}_1(\overline{\rho^p})_{\bar{p}}$ which satisfy Theorem 5.8.

Definition 5.10. For $\rho^p \in \mathscr{M}_p^{\mathscr{C}_r}$ and $j, k \in L$, let

$$C^{jk}(\rho^p): \mathscr{V}_j(\rho^p)_k \to \mathscr{V}_k(\overline{\rho^p})_j \tag{5.59}$$

be the antilinear operator defined by

$$C^{jk}(\rho^p) V^{jk}(\rho^p) = V^{jk}(\rho^p)^* j(\Gamma_{\rho^p \circ \overline{\rho^p}, 1}), \qquad V^{jk}(\rho^p) \in \mathscr{V}_j(\rho^p)_k. \tag{5.60}$$

□

One easily derives from the intertwining properties of (5.60) that $C^{jk}(\rho^p)\, V^{jk}(\rho^p) \in \mathscr{V}_k(\overline{\rho^p})_j$. We often use the notation

$$\bar{V}^{kj}(\overline{\rho^p}) = C^{jk}(\rho^p) V^{jk}(\rho^p). \tag{5.61}$$

With respect to orthonormal bases $\{V^{jk}_\alpha(\rho^p)\}^{N^k_{jp}}_{\alpha=1}$, $\{V^{kj}_\beta(\overline{\rho^p})\}^{N^j_{k\bar{p}}}_{\beta=1}$ in $\mathscr{V}_j(\rho^p)_k$ and $\mathscr{V}_k(\overline{\rho^p})_j$, the matrix $C = (C_{\alpha\beta})$ describing $C^{jk}(\rho^p)$ is given by

$$\bar{V}^{kj}_\alpha(\overline{\rho^p}) = \sum_{\beta=1}^{N^j_{k\bar{p}}} C_{\alpha\beta} V^{kj}_\beta(\overline{\rho^p}) \qquad \alpha = 1, \ldots N^k_{jp}, \tag{5.62}$$

$$C_{\alpha\beta} = \langle V^{kj}_\beta(\overline{\rho^p}); \bar{V}^{kj}_\alpha(\overline{\rho^p})\rangle \qquad \alpha = 1, \ldots N^k_{jp}, \qquad \beta = 1, \ldots N^j_{k\bar{p}}. \tag{5.63}$$

Lemma 5.11. (i) *The antilinear operator* $C^{jk}(\rho^p)$ *satisfies*

$$C^{kj}(\overline{\rho^p}) \cdot C^{jk}(\rho^p) = |\lambda_p| \cdot \mathbb{1}|_{\mathscr{V}_j(\rho^p)_k}, \tag{5.64}$$

$$C^{jk}(\rho^p)\cdot C^{kj}(\overline{\rho^p}) = |\lambda_p|\cdot \mathbb{1}|_{\mathscr{V}_k(\overline{\rho^p})_j}, \tag{5.65}$$

and hence is an anti-isomorphism from $\mathscr{V}_j(\rho^p)_k$ onto $\mathscr{V}_k(\overline{\rho^p})_j$. In particular,

$$N_{jp}^k = N_{k\bar{p}}^j \tag{5.66}$$

holds.

(ii) *The matrix elements $C_{\alpha\beta}$ of $C^{jk}(\rho^p)$ with respect to orthonormal bases $\{V_\alpha^{jk}(\rho^p)\}_{\alpha=1}^{N_{jp}^k}$, $\{V_\beta^{kj}(\overline{\rho^p})\}_{\beta=1}^{N_{k\bar{p}}^j}$ in $\mathscr{V}_j(\rho^p)_k$ and $\mathscr{V}_k(\overline{\rho^p})_j$ are given by*

$$C_{\alpha\beta} = \check{F}(j,p,\bar{p},j)_{1\,1\,1}^{k\alpha\beta}. \tag{5.67}$$

Moreover,

$$0 < \langle \bar{V}_\alpha^{kj}(\overline{\rho^p}); \bar{V}_\alpha^{kj}(\overline{\rho^p})\rangle = \sum_\gamma \hat{F}(j,p,\bar{p},j)_{k\alpha\gamma}^{1\,1\,1}\check{F}(j,p,\bar{p},j)_{1\,1\,1}^{k\alpha\gamma}, \tag{5.68}$$

$$\sum_\gamma \langle V_\alpha^{kj}(\overline{\rho^p}); \bar{V}_\gamma^{kj}(\overline{\rho^p})\rangle\langle \bar{V}_\gamma^{kj}(\overline{\rho^p}); V_\beta^{kj}(\overline{\rho^p})\rangle$$

$$= \sum_\gamma \check{F}(j,p,\bar{p},j)_{1\,1\,1}^{k\gamma\alpha}\hat{F}(j,p,\bar{p},j)_{k\gamma\beta}^{1\,1\,1}. \tag{5.69}$$

□

Proof. (i) We prove only Eq. (5.64); Eq. (5.65) then follows by exchanging p with $\bar{p}$ and using Eq. (5.58). Since

$$C^{kj}(\overline{\rho^p})C^{jk}(\rho^p)V_\alpha^{jk}(\rho^p) = C^{kj}(\overline{\rho^p})[V_\alpha^{jk}(\rho^p)^* j(\Gamma_{\rho^p\circ\overline{\rho^p},1})]$$

$$= j(\Gamma_{\rho^p\circ\overline{\rho^p},1}^*)V_\alpha^{jk}(\rho^p)k(\Gamma_{\overline{\rho^p}\circ\rho^p,1})$$

$$= j\big(\Gamma_{\rho^p\circ\overline{\rho^p},1}^*\,\rho^p(\Gamma_{\overline{\rho^p}\circ\rho^p,1})\big)V_\alpha^{jk}(\rho^p),$$

the result follows from Eq. (5.39).

(ii) Let $\{V_\alpha^{jk}(\rho^p)\}_{\alpha=1}^{N_{jp}^k}, \{V_\beta^{kj}(\overline{\rho^p})\}_{\beta=1}^{N_{k\bar{p}}^j}$ be orthonormal bases in $\mathscr{V}_j(\rho^p)_k$, $\mathscr{V}_k(\overline{\rho^p})_j$. We show that

$$\langle \bar{V}_\alpha^{kj}(\overline{\rho^p}); V_\beta^{kj}(\overline{\rho^p})\rangle = \hat{F}(j,p,\bar{p},j)_{k\alpha\beta}^{1\,1\,1} \tag{5.70}$$

$$\langle V_\beta^{kj}(\overline{\rho^p}); \bar{V}_\alpha^{kj}(\overline{\rho^p})\rangle = \check{F}(j,p,\bar{p},j)_{1\,1\,1}^{k\alpha\beta}. \tag{5.71}$$

These two equations imply (5.67)–(5.69). Equation (5.71) follows from (5.70) by taking complex conjugates,

$$\overline{\langle \bar{V}_\alpha^{kj}(\overline{\rho^p}); V_\beta^{kj}(\overline{\rho^p})\rangle} = \overline{\hat{F}(j,p,\bar{p},j)_{k\alpha\beta}^{1\,1\,1}},$$

since then, by the properties of the scalar product and Eq. (4.95),

$$\langle V^{kj}_{\beta}(\overline{\rho^p}); \bar{V}^{kj}_{\alpha}(\overline{\rho^p})\rangle = \check{F}(j,p,\bar{p},j)^{k\alpha\beta}_{111}\,.$$

The scalar product (5.70) reads,

$$\begin{aligned}&\big(V^{jk}_{\alpha}(\rho^p)^{*} j(\Gamma_{\rho^p\circ\overline{\rho^p},1})\big)^{*}\, V^{kj}_{\beta}(\overline{\rho^p})\\ &\qquad = j(\Gamma^{*}_{\rho^p\circ\overline{\rho^p},1})\, V^{jk}_{\alpha}(\rho^p)\, V^{kj}_{\beta}(\overline{\rho^p})\end{aligned} \tag{5.72}$$

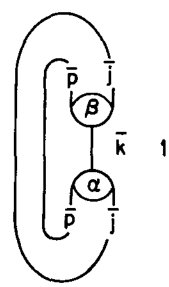
and since

$$V^{jk}_{\alpha}(\rho^p)\, V^{kj}_{\beta}(\overline{\rho^p}) = \sum_{l,\gamma,\delta} \hat{F}(j,p,\bar{p},j)^{l\gamma\delta}_{k\alpha\beta}\, j\big(\Gamma_{\rho^p\circ\overline{\rho^p},\rho^l}(\gamma)\big)\, V^{jj}_{\delta}(\rho^l), \tag{5.73}$$

Eq. (5.70) follows from (5.73) by left multiplication with $j(\Gamma^{*}_{\rho^p\circ\overline{\rho^p},1})$. This completes the proof of Lemma 5.11. ∎

Lemma 5.12. *The following equation holds.*

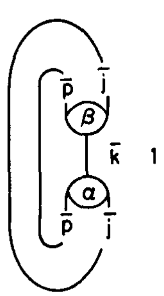
$$\sum_{\gamma} \hat{F}(j,p,\bar{p},j)^{k\gamma\alpha}_{111}\, \check{F}(j,p,\bar{p},j)^{111}_{k\gamma\beta} =$$



(5.74)

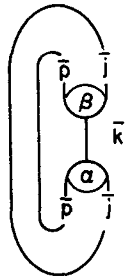
□

Proof. The right hand side of Eq. (5.74) is given by



$$= \sum_{\substack{m,n,\rho,\eta,\\ \sigma,\gamma,\xi,\mu,\varepsilon}} \check{F}(1,j,\bar{j},1)^{m\rho\eta}_{111}\, \check{F}(m,p,\bar{p},m)^{n\sigma\gamma}_{111}\, \hat{F}(n,\bar{p},\bar{j},1)^{\bar{k}\alpha\xi}_{m\gamma\eta}$$
$$\check{F}(n,\bar{p},\bar{j},1)^{m\mu\varepsilon}_{\bar{k}\beta\xi}\, \hat{F}(m,p,\bar{p},m)^{111}_{n\sigma\mu}\, \hat{F}(1,j,\bar{j},1)^{111}_{m\rho\varepsilon}\,.$$

Using the normalization of the fusion matrices on the vacuum sector, Eqs. (4.78) and (4.96), we obtain



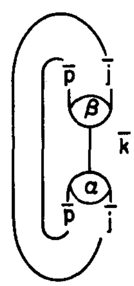
$$1 = \sum_{\substack{m,n,\rho,\eta,\\ \sigma,\gamma,\xi,\mu,\varepsilon}} \delta_{mj}\delta_{\rho 1}\delta_{\eta 1}\check{F}(m,p,\bar{p},m)^{n\sigma\gamma}_{111}\delta_{\bar{k}\bar{n}}\delta_{jm}\delta_{\xi 1}\delta_{\eta 1}\delta_{\alpha\gamma}$$

$$\delta_{\bar{n}\bar{k}}\delta_{mj}\delta_{\varepsilon 1}\delta_{\xi 1}\delta_{\beta\mu}\hat{F}(m,p,\bar{p},m)^{111}_{n\sigma\mu}\delta_{mj}\delta_{\rho 1}\delta_{\varepsilon 1}$$

$$= \sum_{\sigma} \check{F}(j,p,\bar{p},j)^{k\sigma\alpha}_{111}\hat{F}(j,p,\bar{p},j)^{111}_{k\sigma\beta},$$

and this completes the proof. ∎

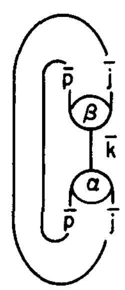
The next lemma, which is a straightforward adaptation of a result of Doplicher, Haag and Roberts ([17, Proposition 6.5; Proposition 6.6]) to the present setting, identifies the explicit value of the graph



in terms of the statistics parameters of the corresponding sectors.

Lemma 5.13.

(i)



$$1 = \frac{|\lambda_{\bar{p}}||\lambda_{\bar{j}}|}{|\lambda_{\bar{k}}|}\cdot\delta_{\alpha\beta}. \tag{5.75}$$

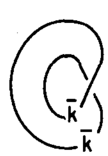
(ii)

$$e^{2\pi i(\theta_{\bar{p},p}+\theta_{\bar{j},j}-\theta_{\bar{k},k})} = e^{2\pi i(s_p+s_j-s_k)}. \tag{5.76}$$

□

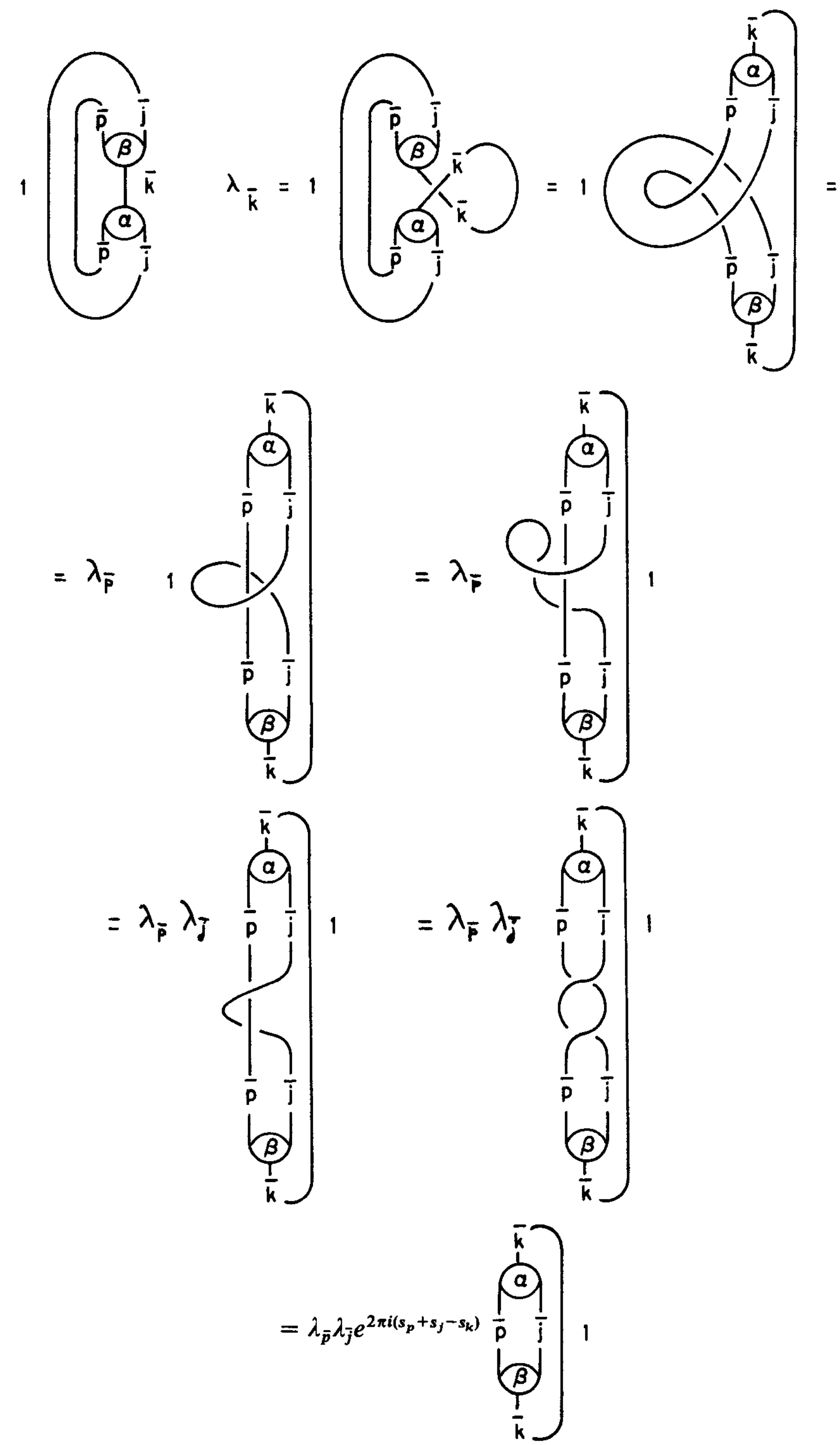
Proof. If we multiply the left hand side of Eq. (5.75) by

$$\lambda_{\bar{k}} =$$

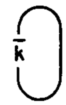


$$1 \tag{5.77}$$

and use the invariance of graphical equations under Reidemeister moves of type II and III, as well as Lemma 5.1, we obtain



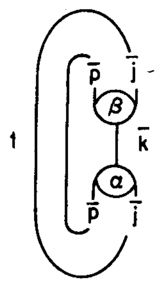
(5.78)

$$= \lambda_{\bar{p}}\lambda_{\bar{j}}e^{2\pi i(s_p+s_j-s_k)}\delta_{\alpha\beta}$$ 

(5.79)

$$= \lambda_{\bar{p}}\lambda_{\bar{j}}e^{2\pi i(s_p+s_j-s_k)}\delta_{\alpha\beta}\,.$$

Equation (5.78) follows by Theorem 4.8 and Eq. (5.79) by Eq. (4.98). Summing up, we obtain



$$= \frac{\lambda_{\bar{p}}\lambda_{\bar{j}}}{\lambda_{\bar{k}}}e^{2\pi i(s_p+s_j-s_k)}\delta_{\alpha\beta}\,. \tag{5.80}$$

Setting $\alpha = \beta$ in (5.80) and using (5.69), we obtain

$$\frac{\lambda_{\bar{p}}\lambda_{\bar{j}}}{\lambda_{\bar{k}}}e^{2\pi i(s_p+s_j-s_k)} = \sum_{\gamma} |\langle V_\alpha^{kj}(\overline{\rho^p}); \bar{V}_\gamma^{kj}(\overline{\rho^p})\rangle|^2 > 0\,. \tag{5.81}$$

This last equation implies Eqs. (5.75) and (5.76). ■

Remark 5.14. Equation (5.76) applied to the case $\bar{j} = p$, $\bar{k} = 1$ implies that

$$e^{4\pi i\theta_{\bar{p},p}} = e^{2\pi i(s_p+s_{\bar{p}})} \tag{5.83}$$

or

$$\theta_{\bar{p},p} = \frac{1}{2}(s_p + s_{\bar{p}})\left(\mathrm{mod}\,\frac{1}{2}\mathbb{Z}\right). \tag{5.83}$$

This is again the weak form of the spin and statistics connection obtained at the end of Sec. 4.2. □

Theorem 5.15. (i) *With respect to orthonormal bases* $\{V_\alpha^{jk}(\rho^p)\}_{\alpha=1}^{N_{jp}^k}$, $\{V_\beta^{kj}(\overline{\rho^p})\}_{\beta=1}^{N_{k\bar{p}}^j}$ *of* $\mathscr{V}_j(\rho^p)_k$ *and* $\mathscr{V}_k(\overline{\rho^p})_j$ *the matrix* $C = (C_{\alpha\beta})$ *of* $C^{jk}(\rho^p)$ *introduced in Eqs.* (5.62) *and* (5.63) *satisfies*

$$CC^* = C^*C = \frac{|\lambda_p||\lambda_j|}{|\lambda_k|}\mathbb{1}\,. \tag{5.84}$$

(ii) *The vectors* $\bar{V}_\alpha^{kj}(\overline{\rho^p})$ *are orthogonal in* $\mathscr{V}_k(\overline{\rho^p})_j$ *and have norm* $\nu(k \circ \overline{\rho^p}; j)$ *equal to*

$$v(k \circ \overline{\rho^p}; j)^2 = \langle \overline{V}_\alpha^{kj}(\overline{\rho^p}); \overline{V}_\alpha^{kj}(\overline{\rho^p}) \rangle$$
$$= \frac{|\lambda_p||\lambda_j|}{|\lambda_k|}, \tag{5.85}$$

for all $\alpha = 1, \dots N_{jp}^k$.

Proof. (i) Since, by Lemma 5.11, $C = (C_{\alpha\beta})$ has matrix elements given by

$$C_{\alpha\beta} = \check{F}(j, p, \overline{p}, j)_{1\,1\,1}^{k\alpha\beta}, \tag{5.86}$$

it follows that

$$(C^* C)_{\alpha\beta} = \sum_\gamma \overline{C}_{\gamma\alpha} C_{\gamma\beta}$$
$$= \sum_\gamma \overline{\check{F}(j, p, \overline{p}, j)_{1\,1\,1}^{k\gamma\alpha}}\, \check{F}(j, p, \overline{p}, j)_{1\,1\,1}^{k\gamma\beta}$$
$$= \sum_\gamma \hat{F}(j, p, \overline{p}, j)_{k\gamma\alpha}^{1\,1\,1}\, \check{F}(j, p, \overline{p}, j)_{1\,1\,1}^{k\gamma\beta}.$$

Equation (5.84) now follows from Lemmas 5.12 and 5.13.

(ii) Orthogonality of the vectors $\overline{V}_\alpha^{kj}(\overline{\rho^p})$, $\alpha = 1, \dots N_{jp}^k$ is clear, since C is unitary up to a constant factor. The norms (5.85) are easily computed by using Eqs. (5.68) and (5.84). This completes the proof of Theorem 5.15. ■

Let us now define a new antilinear operator, $\tilde{C}^{jk}(\rho^p)$, from $\mathscr{V}_j(\rho^p)_k$ onto $\mathscr{V}_k(\overline{\rho^p})_j$ by rescaling $C^{jk}(\rho^p)$:

$$\tilde{C}^{jk}(\rho^p) V_\alpha^{jk}(\rho^p) := \tilde{V}_\alpha^{kj}(\overline{\rho^p}), \tag{5.87}$$

where

$$\tilde{V}_\alpha^{kj}(\overline{\rho^p}) := \frac{1}{v(k \circ \overline{\rho^p}; j)} C^{jk}(\rho^p) V_\alpha^{jk}(\rho^p), \tag{5.88}$$

and $\{V_\alpha^{jk}(\rho^p)\}_{\alpha=1}^{N_{jp}^k}$ is an orthonormal basis of $\mathscr{V}_j(\rho^p)_k$. The properties of $\tilde{C}^{jk}(\rho^p)$ are given by the following theorem.

Theorem 5.16. (i) *The antilinear operator*

$$\tilde{C}^{jk}(\rho^p) : \mathscr{V}_j(\rho^p)_k \to \mathscr{V}_k(\overline{\rho^p})_j$$

is antiunitary and satisfies

$$\tilde{C}^{kj}(\overline{\rho^p}) \tilde{C}^{jk}(\rho^p) = \mathbb{1}|_{\mathscr{V}_j(\rho^p)_k} \tag{5.89}$$

$$\widetilde{C}^{jk}(\rho^p)\widetilde{C}^{kj}(\overline{\rho^p}) = \mathbb{1}|_{\mathscr{V}_k(\overline{\rho^p})_j}. \tag{5.90}$$

(ii) *If* $\{V_\alpha^{jk}(\rho^p)\}_{\alpha=1}^{N_{jp}^k}$ *is an orthonormal basis in* $\mathscr{V}_j(\rho^p)_k$, *then* $\{\widetilde{V}_\alpha^{kj}(\overline{\rho^p})\}_{\alpha=1}^{N_{k\bar{p}}^j}$ *is an orthonormal basis in* $\mathscr{V}_k(\overline{\rho^p})_j$. *With respect to these bases,*

$$\hat{F}(j,p,\bar{p},j)_{k\alpha\beta}^{111} = \check{F}(j,p,\bar{p},j)_{111}^{k\alpha\beta} = \frac{|\lambda_p|^{1/2}|\lambda_j|^{1/2}}{|\lambda_k|^{1/2}}\delta_{\alpha\beta}. \tag{5.91}$$

(iii) *Let* $\rho^p \circ \rho^q \in \mathscr{M}^{\#}_{p\times q}$, $\# = I, II$ *be morphisms localized in spacelike cones* $\mathscr{C}_p$ *and* $\mathscr{C}_q$, *spacelike separated from each other and such that* as $\rho^p >$ as ρ^q. *If*

$$V_\alpha^{jk}(\rho^p)V_\beta^{km}(\rho^p) = \sum_{l,\gamma,\delta} R^+(j,p,q,m)_{k\alpha\beta}^{l\gamma\delta} V_\gamma^{jl}(\rho^p)V_\delta^{lm}(\rho^p) \tag{5.92}$$

holds for orthonormal bases $\{V_\alpha^{jk}(\rho^p)\}_{\alpha=1}^{N_{jp}^k}$, $\{V_\beta^{km}(\rho^q)\}_{\beta=1}^{N_{kq}^m}$, $\{V_\gamma^{jl}(\rho^q)\}_{\gamma=1}^{N_{jq}^l}$ *and* $\{V_\delta^{lm}(\rho^p)\}_{\delta=1}^{N_{lp}^m}$, *then the statistics matrix* $R^-(m,\bar{q},\bar{p},j)_{k\beta\alpha}^{l\delta\gamma}$ *with respect to the orthonormal bases* $\{\widetilde{V}_\alpha^{kj}(\overline{\rho^p})\}_{\alpha=1}^{N_{k\bar{p}}^j}$, $\{\widetilde{V}_\beta^{mk}(\overline{\rho^q})\}_{\beta=1}^{N_{m\bar{q}}^k}$, $\{\widetilde{V}_\gamma^{lj}(\overline{\rho^q})\}_{\gamma=1}^{N_{l\bar{q}}^j}$ *and* $\{\widetilde{V}_\delta^{ml}(\overline{\rho^p})\}_{\delta=1}^{N_{m\bar{p}}^l}$ *is given by*

$$R^-(m,\bar{q},\bar{p},j)_{k\beta\alpha}^{l\delta\gamma} = \overline{R^+(j,p,q,m)_{k\alpha\beta}^{l\gamma\delta}}. \tag{5.93}$$

□

Remark 5.17. Equation (5.93) proves Lemma 4.6 (iii). □

Proof. (i), (ii). Multiplying the orthogonal vectors $\widetilde{V}_\alpha^{kj}(\overline{\rho^p})$ by their inverse norm, we obtain an orthonormal basis $\{\widetilde{V}_\alpha^{kj}(\overline{\rho^p})\}_{\alpha=1}^{N_{k\bar{p}}^j}$ of $\mathscr{V}_k(\overline{\rho^p})_j$. This means that $\widetilde{C}^{kj}(\rho^p)$ is antiunitary and Eqs. (5.89) and (5.90) follow. Choosing $V_\alpha^{kj}(\overline{\rho^p}) = \widetilde{V}_\alpha^{kj}(\overline{\rho^p})$, $\alpha = 1, \ldots N_{k\bar{p}}^j$ implies that

$$V_\alpha^{kj}(\overline{\rho^p}) = \frac{1}{v(k\circ\overline{\rho^p};j)} V_\alpha^{jk}(\rho^p)^* j(\Gamma_{\rho^p\circ\overline{\rho^p},1}).$$

Taking adjoints on both sides, we obtain

$$v(k\circ\overline{\rho^p};j)V_\alpha^{kj}(\overline{\rho^p})^* = j(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})V_\alpha^{jk}(\rho^p) \tag{5.94}$$

so that multiplying Eq. (5.94) from the right by $V_\beta^{kj}(\overline{\rho^p})$,

$$\begin{aligned} v(k\circ\overline{\rho^p};j)\delta_{\alpha\beta} &= j(\Gamma^*_{\rho^p\circ\overline{\rho^p},1})V_\alpha^{jk}(\rho^p)V_\beta^{kj}(\overline{\rho^p}) \\ &= \hat{F}(j,p,\bar{p},j)_{k\alpha\beta}^{111}, \end{aligned} \tag{5.95}$$

where we used Eq. (5.70). Finally, Eq. (5.91) follows from Eqs. (5.95) and (5.85).

(iii) If $\overline{\rho^p}$, $\overline{\rho^q}$ are conjugate to ρ^p and ρ^q and localized in the spacelike separated cones $\mathscr{C}_p$ and $\mathscr{C}_q$ respectively, then locality implies that

$$\Gamma_{\rho^p\circ\overline{\rho^p},1}\Gamma_{\rho^q\circ\overline{\rho^q},1} = \Gamma_{\rho^q\circ\overline{\rho^q},1}\Gamma_{\rho^p\circ\overline{\rho^p},1}\,, \tag{5.96}$$

$$\rho^p(\Gamma_{\rho^q\circ\overline{\rho^q},1}) = \Gamma_{\rho^q\circ\overline{\rho^q},1}\,, \tag{5.97}$$

$$\rho^q(\Gamma_{\rho^p\circ\overline{\rho^p},1}) = \Gamma_{\rho^p\circ\overline{\rho^p},1}\,. \tag{5.98}$$

Taking adjoints on both sides of (5.92) we obtain

$$V_\beta^{km}(\rho^q)^* V_\alpha^{jk}(\rho^p)^* = \sum_{l,\gamma,\delta} \overline{R^+(j,p,q,m)_{k\alpha\beta}^{l\gamma\delta}}\, V_\delta^{lm}(\rho^p)^* V_\gamma^{jl}(\rho^q)^*\,. \tag{5.99}$$

If we multiply both sides of Eq. (5.99) from the right by

$$j(\rho^p(\Gamma_{\rho^q\circ\overline{\rho^q},1})\Gamma_{\rho^p\circ\overline{\rho^p},1}) = j(\rho^q(\Gamma_{\rho^p\circ\overline{\rho^p},1})\Gamma_{\rho^q\circ\overline{\rho^q},1}), \tag{5.100}$$

(where we used Eqs. (5.96)–(5.98)), we obtain

$$\bar V_\beta^{mk}(\overline{\rho^q})\bar V_\alpha^{kj}(\overline{\rho^p}) = \sum_{l,\gamma,\delta} \overline{R^+(j,p,q,m)_{k\alpha\beta}^{l\gamma\delta}}\, \bar V_\delta^{ml}(\overline{\rho^p})\bar V_\gamma^{lj}(\overline{\rho^q}).$$

Rewriting this equation in terms of the vectors $\tilde V_\beta^{mk}(\overline{\rho^q})$, $\tilde V_\alpha^{kj}(\overline{\rho^q})$, $\tilde V_\delta^{ml}(\overline{\rho^p})$ and $\tilde V_\gamma^{lj}(\overline{\rho^q})$ (see Eq. (5.87) and 5.88)), we obtain

$$\tilde V_\beta^{mk}(\overline{\rho^q})\tilde V_\alpha^{kj}(\overline{\rho^p}) = \sum_{l,\gamma,\delta} R^-(m,\bar q,\bar p,j)_{k\beta\alpha}^{l\delta\gamma}\, \tilde V_\delta^{ml}(\overline{\rho^p})\tilde V_\gamma^{lj}(\overline{\rho^q}),$$

with

$$R^-(m,\bar q,\bar p,j)_{k\beta\alpha}^{l\delta\gamma} = \frac{v(m\circ\overline{\rho^p};l)v(l\circ\overline{\rho^q};j)}{v(m\circ\overline{\rho^q};k)v(k\circ\overline{\rho^p};j)}\overline{R^+(j,p,q,m)_{k\alpha\beta}^{l\gamma\delta}} \tag{5.101}$$

since as $\overline{\rho^q} <$ as $\overline{\rho^p}$.

It follows from (5.85) that

$$\frac{v(m\circ\overline{\rho^p};l)v(l\circ\overline{\rho^q};j)}{v(m\circ\overline{\rho^q};k)v(k\circ\overline{\rho^p};j)} = 1\,,$$

and Eq. (5.101) reduces to

$$R^-(m,\bar q,\bar p,j)_{k\beta\alpha}^{l\delta\gamma} = \overline{R^+(j,p,q,m)_{k\alpha\beta}^{l\gamma\delta}}\,.$$

which is (5.93). ■

This completes our analysis of statistics and of fusion identities in algebraic quantum field theory.

In the next section we use this formalism to describe the multi-matrix algebras corresponding to the commutant of reducible representations of the observable algebra $\mathscr{A}$. This will establish a precise connection between the theory of superselection sectors and their statistics in algebraic quantum field theory and Jones' theory of towers of algebras [28, 51].

6. Path Algebras in Quantum Field Theory, Mapping Class Groups

6.1. *The commutant of a reducible representation of $\mathscr{A}$*

In this section, we show how the intertwiners $V_\alpha^{kj}(\rho)$ and the braid and fusion matrices may be used to provide a detailed description for the commutant of a product $\rho_m \times \cdots \times \rho_0$ of representations of $\mathscr{A}$. Such commutants play a natural role in the determination of the statistics and internal symmetries of a superselection sector. It is to be expected that a careful analysis of their structure will be essential to the resolution of two outstanding problems which remain unsolved for low-dimensional quantum field theories: the classification of all braid statistics and of internal symmetries compatible with the general principles of relativistic, quantum physics. Some of our results are similar to ones derived by Fredenhagen, Rehren and Schroer in the context of two-dimensional quantum field theories, although our approach is different from the one used in [22]. To avoid technical problems, mainly convergence issues, we sometimes consider only rational theories, i.e., theories having only finitely many inequivalent superselection sectors. The results for theories having an arbitrary number of superselection sectors are expected to be similar.

Let $p_1, p_2, p_3, \ldots$ be a family of representations in L, and let $\rho_1, \rho_2, \rho_3, \ldots$ be morphisms corresponding to the sectors $p_1, p_2, p_3, \ldots$, respectively. For $p_0 \in L$, choose a morphism $\rho_0 \in p_0$ and define

$$M_0^{p_0} = \rho_0(\mathscr{A})' = \{\lambda \cdot \mathbb{1}; \lambda \in \mathbb{C}\}, \tag{6.1}$$

$$M_n^{p_0} = (\rho_0 \circ \rho_1 \circ \cdots \circ \rho_{n-1} \circ \rho_n(\mathscr{A}))'. \tag{6.2}$$

By property (P1) of Sec. 2, each $M_n^{p_0}$ is a full multi-matrix ring, and

$$\mathbb{1} \in M_k^{p_0} \subseteq M_j^{p_0}, \qquad k \leq j, \tag{6.3}$$

for all $j, k \in \{0\} \cup \mathbb{N}$. The fusion matrices $\mathbb{N}_{p_j}, j = 1, 2, 3, \ldots$, adequately describe this chain of inclusions,

$$\{\lambda \cdot \mathbb{1}\} = M_0^{p_0} \subseteq M_1^{p_0} \subseteq \cdots, \tag{6.4}$$

as we now explain. First consider the product $\rho_0 \circ \rho_1$. The decomposition

$$p_0 \times p_1 = p_1 \times p_0 \cong \bigoplus_{l \in L} \bigoplus_{\alpha=1}^{N_{p_1 p_0}^l} l^{(\alpha)}, \tag{6.5}$$

into finitely many, mutually inequivalent, irreducible representations $l \in L$ of $\mathscr{A}$, implies that

$$M_1^{p_0} = \rho_0 \circ \rho_1(\mathscr{A})' \cong \bigoplus_{l \in L} \mathrm{Mat}_{N^l_{p_1 p_0}}(\mathbb{C}), \tag{6.6}$$

with the convention that the direct summand $\mathrm{Mat}_{N^l_{p_1 p_0}}(\mathbb{C})$ is equal to $\{0\}$ if $N^l_{p_1 p_0} = 0$. Proceeding one step further,

$$p_0 \times p_1 \times p_2 = p_2 \times p_1 \times p_0 \cong \bigoplus_{l \in L} \bigoplus_{\alpha=1}^{N^l_{p_1 p_0}} p_2 \times l^{(\alpha)} \tag{6.7}$$

$$\cong \bigoplus_{k \in L} \bigoplus_{l \in L} \bigoplus_{\alpha=1}^{N^k_{p_2 l} N^l_{p_1 p_0}} k^{(\alpha)} \tag{6.8}$$

and introducing the dimension vectors

$$\mu^0 = (\mu^0_l)_{l \in L} = (\delta_{l p_0})_{l \in L}, \tag{6.9}$$

$$\mu^1 = (\mu^1_l)_{l \in L} = (N^1_{p_1 p_0})_{l \in L} = \mathbb{N}_{p_1} \mu^0, \tag{6.10}$$

$$\mu^n = (\mu^n_l)_{l \in L} = \mathbb{N}_{p_n} \mu^{n-1}, \qquad n = 2, 3, \ldots \tag{6.11}$$

we may write

$$M_0^{p_0} \cong \mathbb{C} \tag{6.12}$$

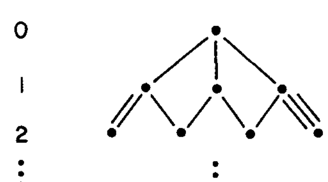
$$M_1^{p_0} \cong \bigoplus_{l \in L} \mathrm{Mat}_{\mu^1_l}(\mathbb{C}) \tag{6.13}$$

$$M_2^{p_0} \cong \bigoplus_{k \in L} \mathrm{Mat}_{\mu^2_k}(\mathbb{C}), \qquad \text{etc.} \tag{6.14}$$

Furthermore, Eq. (6.8) implies that the inclusion $M_1^{p_0} \subseteq M_2^{p_0}$ is given by

$$\bigoplus_{k \in L} \bigoplus_{l \in L} N^k_{p_2 l} \mathrm{Mat}_{\mu^1_l}(\mathbb{C}) \subseteq \bigoplus_{k \in L} \mathrm{Mat}_{\mu^2_k}(\mathbb{C}) \tag{6.15}$$

so that the associated inclusion matrix is the block $\Lambda_{p_2} = (N^k_{p_2 l})$ of $\mathbb{N}_{p_2}$ where $k, l \in L$ are such that $\mu^1_l \neq 0, \mu^2_k \neq 0$. Iterating the preceding steps, we see that the multi-matrix chain (6.3) is fully determined [50, 51] by the unit vector μ^0 and the set of fusion matrices $\{\mathbb{N}_{p_k}\}_{k \in \mathbb{N}}$. These data are conveniently encoded in an (infinite) Bratteli diagram [50, 51]:



(6.16)

Each $n = 0, 1, 2, \ldots$ labels a floor of the diagram, and each floor, n, has vertices $k_n^1, \ldots, k_n^l$, one for each non-vanishing coordinate of the dimension vector μ^n. A vertex k_{n-1} is joined to each vertex k_n by $N_{p_n k_{n-1}}^{k_n}$ edges labelled by triples (k_{n-1}, α_n, k_n), $\alpha_n \in \{1, 2, \ldots, N_{p_n k_{n-1}}^{k_n}\}$. A connected sequence of such edges (l, α, m) determines a path $\omega = (l, \alpha, m) \circ (m, \beta, n) \circ \ldots$ on the Bratteli diagram (6.16). Denote by $\mathbb{C}\Omega$ the complex vector space having as basis the set of monotone-increasing paths ω starting at the zeroth floor of (6.16), and by $\mathbb{C}\Omega_n$ the space having as basis paths of length n (i.e., ending at the n-th floor). As is well known [51, 52, 53], it is possible to construct a model for the chain (6.4) by considering algebras of operators acting on $\mathbb{C}\Omega$. The following result is then obvious.

Lemma 6.1. *Products of intertwiners*

$$V^{1p_0}(\rho_0) V_{\alpha_1}^{p_0 k_1}(\rho_1) V_{\alpha_2}^{k_1 k_2}(\rho_2) \ldots V_{\alpha_n}^{k_{n-1} k_n}(\rho_n) \tag{6.17}$$

are in one-to-one correspondence with the set of monotone-increasing paths $\omega_n = (p_0, \alpha_1, k_1) \circ (k_1, \alpha_2, k_2) \circ \cdots \circ (k_{n-1}, \alpha_n, k_n) \in \mathbb{C}\Omega_n$. We write V_{ω_n} for the product (6.17) and henceforth identify ω_n with V_{ω_n}. The vector spaces $\mathbb{C}\Omega_n$, $\mathbb{C}\Omega$, respectively, carry a natural scalar product in which the V_{ω_n}'s build an orthonormal basis. If we define the operators

$$T_{(\eta_n;\omega_n)} : \mathbb{C}\Omega_n \to \mathbb{C}\Omega_n$$

$$V_{\xi_n} \mapsto \langle V_{\omega_n}; V_{\xi_n} \rangle V_{\eta_n} \tag{6.18}$$

or

$$T_{(\eta_n;\omega_n)} = V_{\eta_n} V_{\omega_n}^*, \tag{6.19}$$

for two paths η_n, ω_n which are always assumed to have the same endpoints, then

$$A_n^{p_0} = \{T_{(\eta_n;\omega_n)}, \eta_n, \omega_n \in \mathbb{C}\Omega_n, \text{ with matching endpoints}\} \tag{6.20}$$

is isomorphic to $M_n^{p_0}$; $T_{(\eta_n;\omega_n)}$ extends in an obvious way to an operator on $\mathbb{C}\Omega$. For additional details, see [51]. □

For rational theories, we define a further multimatrix chain by summing over all possible initial sectors $p_0 \in L$,

$$M_n = \Big(\bigoplus_{p_0 \in L} (\rho_0 \circ \cdots \circ \rho_{n-1} \circ \rho_n(A)) \Big)', \qquad n = 0, 1, 2, \ldots, \tag{6.21}$$

so that we obtain the sequence of inclusions

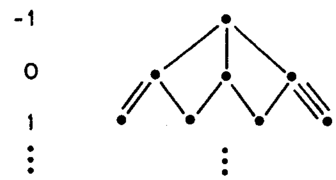
$$\{\lambda \mathbb{1}; \lambda \in \mathbb{C}\} \subseteq M_0 \cong \bigoplus_{q \in L} \mathbb{C} \subseteq M_1 \subseteq M_2 \subseteq \cdots \tag{6.22}$$

described by the dimension vector

$$\tilde{\mu}^0 = (\tilde{\mu}^0_l)_{l \in L}, \tag{6.23}$$

$$\tilde{\mu}^0_l = 1, \qquad \text{for all } l \in L, \tag{6.24}$$

and the inclusion matrices $\mathbb{N}_{p_1}$, $\mathbb{N}_{p_2}$, etc .... The Bratteli diagram of the chain (6.22) is obtained by placing on each floor 0, 1, 2, ... a vertex for each sector $l \in L$ and $N^i_{p_k j}$ edges between the vertex j of the $k-1^{\text{st}}$ floor and the vertex i of the k^{th} floor. Furthermore, one adds at the top of the diagram a -1^{st} floor with a single vertex connected by a single edge to each vertex l of the zeroth floor:



(6.25)

The paths $\omega_n = (-1, 1, k_0) \circ (k_0, \alpha_1, k_1) \circ \cdots \circ (k_{n-1}, \alpha_n, k_n)$ on the diagram (6.25) are in one-to-one correspondence with products of intertwiners (6.17), where the initial sector p_0 is allowed to vary over L,

$$V^{1k_0}(\rho_0) V^{k_0 k_1}_{\alpha_1}(\rho_1) \ldots V^{k_{n-1}k_n}_{\alpha_n}(\rho_n), \qquad \rho_0 \in k_0. \tag{6.26}$$

The remainder of Lemma 6.1 is unchanged, and M_n is isomorphic to the corresponding path algebra A_n.

Before proceeding further, we require the following definition.

Definition 6.2. Let λ_p be the statistics parameter of the sector $p \in L$. Then we may decompose λ_p as follows:

$$\lambda_p = \frac{1}{d(p)} e^{-2\pi i \theta_{p.\bar{p}}}, \tag{6.27}$$

$$|\lambda_p| = \frac{1}{d(p)} > 0. \tag{6.28}$$

The real number $d(p)$ is called the statistical dimension of the sector p. It will also be convenient to define the row vector of statistical dimensions

$$d = (d(p))_{p \in L}. \tag{6.29}$$

□

Lemma 6.3. *The vector of statistical dimensions d is the Perron-Frobenius eigenvector of the fusion matrix $\mathbb{N}_p$ with eigenvalue $d(p)$,*

$$d \cdot \mathbb{N}_p = d(p) \cdot d, \tag{6.30}$$

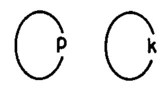
for all $p \in L$. □

Proof. Equation (6.30) is equivalent to

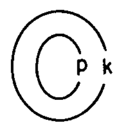
$$\sum_{l \in L} d(l) N_{pk}^l = d(p) d(k). \tag{6.31}$$

Using the graphical formalism of Sec. 5,

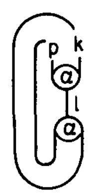
$$d(p)d(k) = \frac{1}{|\lambda_{\bar{p}}|}\frac{1}{|\lambda_k|} \tag{6.32}$$



$$= \frac{1}{|\lambda_p|}\frac{1}{|\lambda_k|}$$



$$= \sum_{\substack{l \in L \\ \alpha = 1, \ldots, N_{pk}^l}} \frac{1}{|\lambda_p|}\frac{1}{|\lambda_k|} \tag{6.33}$$



$$= \sum_{l \in L} d(l) N_{pk}^l, \tag{6.34}$$

where Eq. (6.33) follows from (4.99) and Eq. (6.34) from Lemma 5.13. ■

We now turn to the analysis of the path algebras of Lemma 6.1, for rational theories. Of particular interest to us are the two special cases

(i) $p_n = p$; and
(ii) $p_{2n} = \bar{p}$, $p_{2n+1} = p$, (6.35)

for all $n = 0, 1, 2, \ldots$, where p is an arbitrary sector in L. The importance of these two special cases is due to the following lemma.

Lemma 6.4. (i) *The multimatrix chain*

$$\rho(\mathscr{A})' \subseteq \rho^2(\mathscr{A})' \subseteq \cdots \tag{6.36}$$

associated to (6.35) (i) *by definition* (6.1), (6.2) *carries a unitary representation of the braid groups* B_n, *for* $n = 1, 2, \ldots$ *which is determined by the* $R^{\pm}$*-matrices.*

(ii) *The multimatrix chain*

$$\{\lambda \mathbb{1}; \lambda \in \mathbb{C}\} \subseteq M_0 \subseteq M_1 \subseteq \cdots \tag{6.37}$$

associated to (6.35) (ii) *by definition* (6.21) *is a tower in the sense of Jones. The matrix* $\mathbb{N}_p$ *describes the inclusion* $M_0 \subseteq M_1$ *and the index of this inclusion is*

$$[M_1; M_0] = d(p)^2. \tag{6.38}$$

□

Remark 6.5. Lemma 6.3 identifies the Perron-Frobenius eigenvector of Lemma 2.4 with the vector of statistical dimensions. Lemma 6.4 (ii) is analogous to the result of Longo [31] that the index of the tunnel

$$\mathscr{A}(\mathscr{C})^{-\omega} \supseteq \rho\big(\mathscr{A}(\mathscr{C})^{-\omega}\big) \supseteq \bar{\rho} \circ \rho\big(\mathscr{A}(\mathscr{C})^{-\omega}\big) \supseteq \cdots \tag{6.39}$$

is given by $d(p)^2$. By a well-known theorem of Jones [28], Eq. (6.38) restricts the possible values of $d(p)^2$ to the set $\left\{4\cos^2\left(\dfrac{\pi}{n}\right), n = 3, 4, \ldots\right\} \cup [4, \infty)$. □

Proof. (i) For each $n = 1, 2, 3, \ldots$ we specify a map π sending σ_n to an operator $\pi(\sigma_n)$ in A_n^p which is unitary and such that the braid relations

$$\begin{aligned} \pi(\sigma_i)\pi(\sigma_j) &= \pi(\sigma_j)\pi(\sigma_i), \qquad |i-j| \geq 2, \\ \pi(\sigma_i)\pi(\sigma_{i+1})\pi(\sigma_i) &= \pi(\sigma_{i+1})\pi(\sigma_i)\pi(\sigma_{i+1}) \end{aligned} \tag{6.40}$$

hold. This defines a representation of $B_\infty = \bigcup_{n>0} B_n$ on the multimatrix chain. Let ω_{n-2} denote a path of length $n-2$ on the Bratteli diagram (6.16) and set

$$\begin{aligned} \sigma_n \mapsto \pi(\sigma_n) = \sum_{\substack{k_n, k_{n-1}, \alpha_{n-1}, \alpha_n, \\ k'_{n-1}, \alpha'_{n-1}, \alpha'_n, \omega_{n-2}}} & R^+(k_{n-2}, p, p, k_n)^{k'_{n-1}\alpha'_{n-1}\alpha'_n}_{k_{n-1}\alpha_{n-1}\alpha_n} \\ & \times V_{\omega_{n-2}} V^{k_{n-2}k'_{n-1}}_{\alpha'_{n-1}}(\rho) V^{k'_{n-1}k_n}_{\alpha'_n}(\rho) V^{k_{n-1}k_n}_{\alpha_n}(\rho)^* V^{k_{n-2}k_{n-1}}_{\alpha_{n-1}}(\rho)^* V^*_{\omega_{n-2}}. \end{aligned} \tag{6.41}$$

When applied to a basis vector $\omega_n = \omega_{n-2} \circ (k_{n-2}, \alpha_{n-1}, k_{n-1}) \circ (k_{n-1}, \alpha_{n-1}, k_n)$ of $\mathbb{C}\Omega_n$, the action of (6.41) reduces to

$$\begin{aligned} \pi(\sigma_n) V_{\omega_{n-2}} V^{k_{n-2}k_{n-1}}_{\alpha_{n-1}}(\rho) V^{k_{n-1}k_n}_{\alpha_n}(\rho) = \sum_{k'_{n-1}, \alpha'_{n-1}, \alpha'_n} & R^+(k_{n-2}, p, p, k_n)^{k'_{n-1}\alpha'_{n-1}\alpha'_n}_{k_{n-1}\alpha_{n-1}\alpha_n} \\ & \times V_{\omega_{n-2}} V^{k_{n-2}k'_{n-1}}_{\alpha'_{n-1}}(\rho) V^{k'_{n-1}k_n}_{\alpha'_n}(\rho). \end{aligned} \tag{6.42}$$

Unitarity of $\pi(\sigma_n)$ follows from unitarity of the $R^\pm$-matrices, and the fact that π is a representation of the braid groups, B_n; from the Yang-Baxter equations.

(ii) By Lemma 5.11,

$$N^k_{pl} = N^l_{\bar{p}k} \tag{6.43}$$

so that

$$\mathbb{N}_p = \mathbb{N}^t_{\bar{p}}.$$

Hence the inclusions

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \tag{6.44}$$

are described by the matrices $\mathbb{N}_p$, $\mathbb{N}^t_p$, ... and the dimension vector $\vec{\mu}^0 = (1, 1, \ldots)$ so that (6.44) is a tower. Equation (6.38) follows from Lemma 6.3 and [28]. ■

Remark 6.6. (i) It follows from Eqs. (6.41), (6.42) and Lemma 4.3 that the representation π of B_n defined in Lemma 6.4 (i) coincides with the usual algebraic definition

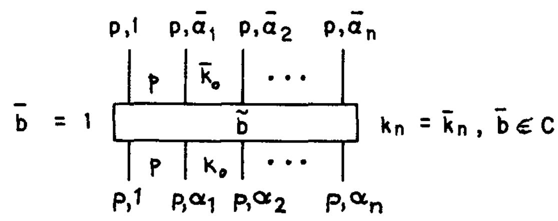
$$\pi \colon \sigma_n \to \rho^{n-1}(\varepsilon^+_{\rho,\rho}). \tag{6.45}$$

(ii) The statement of Lemma 6.4 (i) holds unchanged for a theory having an arbitrary number of superselection sectors. The proof of Lemma 6.4 (i) shows that, for a rational theory, the R^+-matrix elements also define a unitary representation of the braid groups on the multimatrix chain

$$\{\lambda \mathbb{1}; \lambda \in \mathbb{C}\} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \tag{6.46}$$

associated to (6.35) (i) by Definition (6.21). □

In the following, we first concentrate on the multimatrix chain associated to a single representation $p \in L$ (case (i) of (6.35)). Our first goal is to define a trace state on the group algebra of B_n associated to the representation π which coincides with the usual trace state of algebraic field theory. We remark that the graphical diagrams involving $R^{\pm}$-matrix elements acquire a new significance after the proof of Lemma 6.4: if we consider a diagram with n incoming and outgoing strings, fully decorated with labels, such as

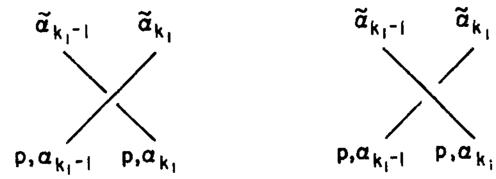


(6.47)

where the box $\tilde{b}$ contains an arbitrary combination of $R^{\pm}$-matrices, then we may assign to $\tilde{b}$ an element b of the braid group B_n such that

$$\bar{b} = \langle V_{\bar{\omega}_n}; \pi(b) V_{\omega_n} \rangle, \tag{6.48}$$

where $\omega_n = (p, \alpha_1, k_1) \circ \cdots \circ (k_{n-1}, \alpha_n, k_n)$ and $\bar{\omega}_n = (p, \bar{\alpha}_1, \bar{k}_1) \circ \cdots \circ (\bar{k}_{n-1}, \bar{\alpha}_n, k_n)$. Conversely, to any $b \in B_n$ there corresponds a diagram $\tilde{b}$ such that Eq. (6.48) is true for any paths ω_n, $\bar{\omega}_n$. We now briefly explain how to reconstruct b from the diagram $\tilde{b}$, the remaining statements are then obvious. Read the diagram $\tilde{b}$ from bottom to top and at the first crossing,



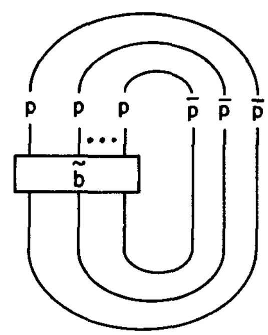
(i) (ii)

Fig. 6.1

assign to Fig. 6.1 (i) and (ii) the braid group generator σ_{k_l}, $\sigma_{k_l}^{-1}$, respectively. Relabel all outgoing lines after the first crossing with dummy indices $\tilde{\alpha}_k$, $k = 0, 1, 2, \ldots$ in increasing order from left to right, as noted in Fig. 6.1 for example, and proceed to the next crossing. Multiplying the $\sigma_{k_l}^{\pm 1}$'s, $l = 1, \ldots r$, we obtain an element $b = \sigma_{k_1}^{\pm 1} \sigma_{k_2}^{\pm 1} \ldots \sigma_{k_n}^{\pm 1}$ such that Eq. (6.48) holds.

The preceding remark allows us to define immediately a functional on the braid group B_n: for $b \in B_n$, take the diagram $\tilde{b}$ associated to b and close the braids (using the conventions of Sec. 5) as follows:

$\varphi_n(b) = 1$



1 (6.49)

Equation (6.49) extends by linearity to the group algebra of B_n.

Lemma 6.7. *The functionals* φ_n: $\mathbb{C}[B_n] \to \mathbb{C}$ *satisfy*

(i) $\varphi_{n+1}(b) = \varphi_n(b)$ *if* $b \in B_n$.

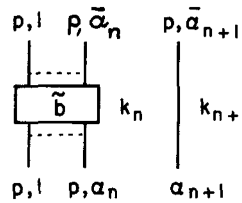
(ii) $\varphi_n(b\sigma_n) = \lambda_p \varphi_n(b)$,
$\varphi_n(b\sigma_n^{-1}) = \bar{\lambda}_p \varphi_n(b)$ *for* $b \in B_{n-1}$.

(iii) $\varphi_n(b_1 b_2) = \varphi_n(b_2 b_1) \forall b_1, b_2 \in B_n$.

(iv) $\varphi_n(b_1 b_2) = \varphi_n(b_1)\varphi_n(b_2)$, *if* $b_1 \in B_k$ *and* $b_2 \in B_{n-k}$,
where B_{n-k} *denotes words in the generators* $\{\sigma_{k+1}, \ldots, \sigma_n\}$ *of* B_n.

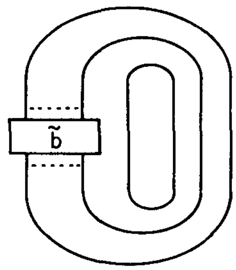
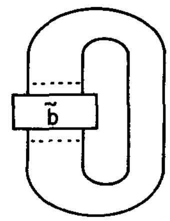
(v) $\varphi_n(1) = 1$. □

Proof. (i) If $b \in B_n$, then

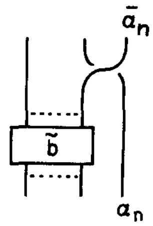


(6.50)

is the diagram for b in B_{n+1}. It follows that

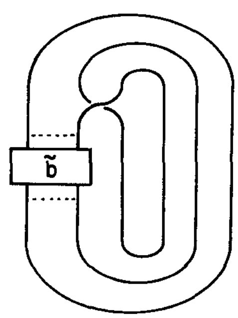
$\varphi_{n+1}(b) =$  $=$  $= \varphi_n(b)$ (6.51)

(ii) The element $b\sigma_n$, $b \in B_{n-1}$ has the associated diagram



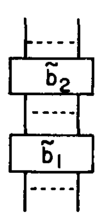
(6.52)

so that

$\varphi_n(b\sigma_n) =$  (6.53)

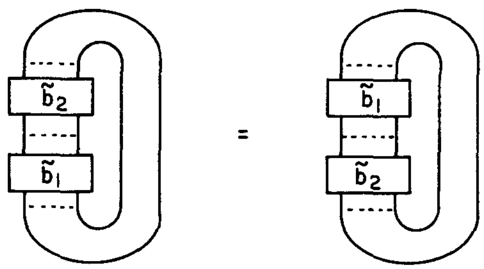
and the statement follows by Lemma 5.1. The second identity in (ii) is shown in the same way.

(iii) Let b_1 and b_2 be any elements of B_n: the diagram of $b_1 b_2$ is



(6.54)

and we want to show that

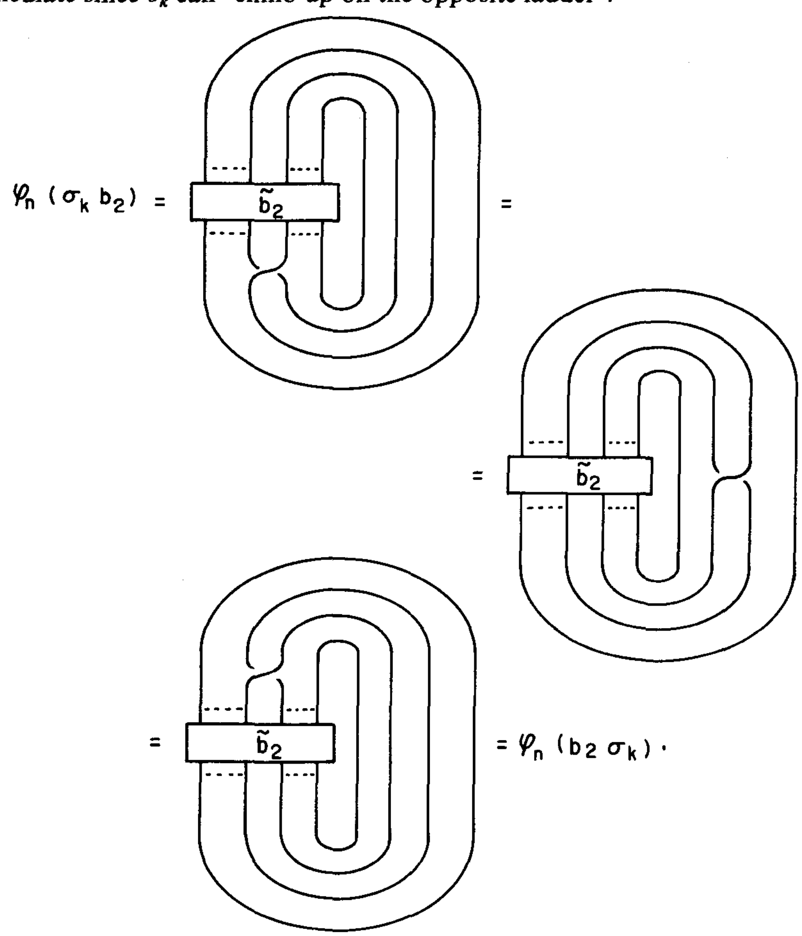


(6.55)

Since b_1 and b_2 may be fully decomposed into a product of σ_k's, it is sufficient to show that

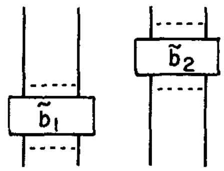
$$\varphi_n(\sigma_k b_2) = \varphi_n(b_2 \sigma_k), \qquad \forall b_2 \in B_n . \tag{6.56}$$

The assumption will then follow by repeated use of Eq. (6.56). But Eq. (6.56) is immediate since σ_k can "climb up on the opposite ladder":



(6.57)

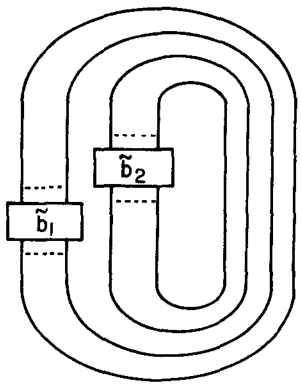
(iv) The proof of this property is particularly simple: if $b_1 \in B_k$ and $b_2 \in B_{n-k}$ then the diagram of $b_1 b_2$ is



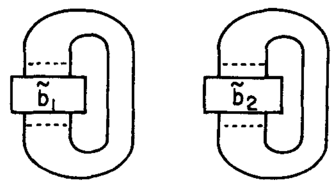
(6.58)

so that

$\varphi_n(b_1, b_2) =$



$=$



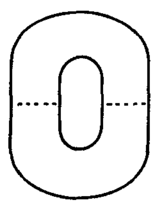
(6.59)

$= \varphi_n(b_1) \cdot \varphi_n(b_2)$

where (6.59) follows by repeated use of Reidemeister moves of type II and III; see Eqs. (5.20)–(5.25).

(v) is immediate, since

$\varphi(1) =$



$= 1$ (6.60)

by repeated use of (4.98). ∎

Remark 6.8. It follows from Lemma 6.7 that the collection of φ_n's defines a trace state φ on $B_\infty = \bigcup_{n>0} B_n$. This state is called a *Markov trace*, because it enjoys property (ii) of our last lemma. We use the notation $\varphi = \mathrm{tr}_M$ for the Markov trace on B_∞. □

Next we show that the trace φ_n can be extended to the whole multi-matrix algebra $A_n^p \cong \bigoplus_{k \in L} \mathrm{Mat}_{\mu_k^n}(\mathbb{C})$. A trace on the multi-matrix algebra A_n^p is fully determined by its trace vector $t^n = (t_k^n)_{k \in L}$ where t_k^n is the trace of a minimal idempotent in $\mathrm{Mat}_{\mu_k^n}(\mathbb{C})$. We will explicitly compute the components t_k^n, $k \in L$, of the vector t^n and show that they are strictly positive. This implies faithfulness of the trace φ_n as a state on A_n^p.

The path space $\mathbb{C}\Omega_n$ is isomorphic to a direct sum of tensor products

$$\mathbb{C}\Omega_n \cong \bigoplus_{k_1, \ldots, k_n} \mathbb{C}^{N_{p_1}^p} \otimes \mathbb{C}^{N_{pp}^{k_1}} \otimes \cdots \otimes \mathbb{C}^{N_{pk_{l-1}}^{k_l}} \otimes \cdots \otimes \mathbb{C}^{N_{pk_{n-1}}^{k_n}}. \tag{6.61}$$

A basis for $\mathbb{C}\Omega_n$ is given by the vectors

$$V^{1p}(\rho)V_{\alpha_1}^{pk_1}(\rho)V_{\alpha_2}^{k_1k_2}(\rho)\dots V_{\alpha_l}^{k_{l-1}k_l}(\rho)\dots V_{\alpha_n}^{k_{n-1}k_n}(\rho).$$

If we identify the orthonormal basis of the vector space $\mathbb{C}^{N_{pk_{l-1}}^{k_l}}$ with the set $\{V_{\alpha_l}^{k_{l-1}k_l}\}_{\alpha_l=1}^{N_{pk_{l-1}}^{k_l}}$ we may define two linear operators $\hat{F}$ and $\check{F}$ which act on $\mathbb{C}^{N_{pk_{l-1}}^{k_l}}$ by setting

$$\begin{aligned} \check{F}|_{\mathbb{C}^{N_{pk_{l-1}}^{k_l}}} &= \sum_{\alpha_l,\beta_l} \check{F}(k_{l-1},p,\bar{p},k_{l-1})_{1\,1\,1}^{k_l\beta_l\alpha_l} T_{(\beta_l;\alpha_l)} \\ \hat{F}|_{\mathbb{C}^{N_{pk_{l-1}}^{k_l}}} &= \sum_{\alpha_l,\beta_l} \hat{F}(k_{l-1},p,\bar{p},k_{l-1})^{1\,1\,1}_{k_l\alpha_l\beta_l} T_{(\beta_l;\alpha_l)} \end{aligned} \tag{6.62}$$

where

$$T_{(\beta_l;\alpha_l)}\colon \mathbb{C}^{N_{pk_{l-1}}^{k_l}} \mapsto \mathbb{C}^{N_{pk_{l-1}}^{k_l}}$$

$$V_\gamma^{k_{l-1}k_l}(\rho) \mapsto \langle V_{\alpha_l}^{k_{l-1}k_l}(\rho); V_\gamma^{k_{l-1}k_l}(\rho)\rangle V_{\beta_l}^{k_{l-1}k_l}(\rho).$$

Using Eqs. (6.61) and (6.62) we construct two operators $\hat{F}$ and $\check{F}$ in the path algebra A_n^p as follows:

$$\begin{aligned} \check{F} &= \sum_{k_1,\dots,k_n} \check{F}|_{\mathbb{C}^{N_{1p}^p}} \otimes \check{F}|_{\mathbb{C}^{N_{pp}^{k_1}}} \otimes \cdots \otimes \check{F}|_{\mathbb{C}^{N_{pk_{n-1}}^{k_n}}} \\ \hat{F} &= \sum_{k_1,\dots,k_n} \hat{F}|_{\mathbb{C}^{N_{1p}^p}} \otimes \hat{F}|_{\mathbb{C}^{N_{pp}^{k_1}}} \otimes \cdots \otimes \hat{F}|_{\mathbb{C}^{N_{pk_{n-1}}^{k_n}}}. \end{aligned} \tag{6.63}$$

Lemma 6.9. *Let b_n be an element of B_n. Then the Markov trace of Lemma 6.7 on the group algebra of B_n is given by*

$$\mathrm{tr}_M(b_n) = \mathrm{tr}(\hat{F}\cdot\pi(b_n)\cdot\check{F}), \tag{6.64}$$

where $\mathrm{tr}(\cdot)$ is the standard trace on the algebra $A_n^p = \bigoplus_{k\in I_t} \mathrm{Mat}_{\mu_k^n}(\mathbb{C})$. Clearly, the Markov trace tr_M extends to the whole algebra A_n^p. □

Proof. This lemma is nothing more than a reinterpretation of the diagram (6.49) defining the trace: each curved leg

(6.65)

corresponding to an operator $\hat{F}$ on the appropriate subspace $\mathbb{C}^{N_{pk_{l-1}}^{k_l}}$ and each

(6.66)

to an operator $\check{F}$. ■

We now turn to the computation of the Markov trace of a minimal idempotent of A_n^p. Let $\omega_n = (p, \gamma_1, l_1) \circ (l_1, \gamma_2, l_2) \circ \cdots \circ (l_{n-1}, \gamma_n, l_n)$ so that

$$\begin{aligned} T_{(\omega_n;\omega_n)} &= V_{\omega_n} V_{\omega_n}^*, \\ V_{\omega_n} &= V^{1p}(\rho) V_{\gamma_1}^{pl_1}(\rho) V_{\gamma_2}^{l_1 l_2}(\rho) \ldots V_{\gamma_n}^{l_{n-1} l_n}(\rho) \end{aligned} \tag{6.67}$$

is a minimal idempotent of A_n^p in $\mathrm{Mat}_{\mu_{l_n}^n}(\mathbb{C})$. Then

$$t_{l_n}^n = \mathrm{tr}_M(T_{(\omega_n;\omega_n)}) = \mathrm{tr}(\hat{F} V_{\omega_n} V_{\omega_n}^* \check{F}). \tag{6.68}$$

On A_n^p, we have

$$\begin{aligned} \hat{F} &= \sum_{\underline{k}} \sum_{\substack{\alpha(\underline{k}) \\ \beta(\underline{k})}} \hat{F}_{\underline{k}\alpha(\underline{k})\beta(\underline{k})} V(\underline{k}, \alpha(\underline{k})) V(\underline{k}, \beta(\underline{k}))^*, \\ \check{F} &= \sum_{\underline{k}} \sum_{\substack{\alpha(\underline{k}) \\ \beta(\underline{k})}} \check{F}_{\underline{k}\beta(\underline{k})\alpha(\underline{k})} V(\underline{k}, \alpha(\underline{k})) V(\underline{k}, \beta(\underline{k}))^* \end{aligned} \tag{6.69}$$

where we use the notation

$$\underline{k} = (k_1, \ldots, k_n), \qquad \underline{l} = (l_1, \ldots, l_n),$$

$$\alpha(\underline{k}) = (\alpha_1(\underline{k}), \ldots, \alpha_n(\underline{k})),$$

$$\beta(\underline{k}) = (\beta_1(\underline{k}), \ldots, \beta_n(\underline{k})), \qquad \gamma = (\gamma_1, \ldots, \gamma_n)$$

$$\hat{F}_{\underline{k}\alpha(\underline{k})\beta(\underline{k})} = \hat{F}(p, p, \bar{p}, p)_{k_1\alpha_1\beta_1}^{111} \hat{F}(k_1, p, \bar{p}, k_1)_{k_2\alpha_2\beta_2}^{111} \times \cdots \times \hat{F}(k_{n-1}, p, \bar{p}, k_{n-1})_{k_n\alpha_n\beta_n}^{111} \tag{6.70}$$

$$\check{F}_{\underline{k}\beta(\underline{k})\alpha(\underline{k})} = \check{F}(p, p, \bar{p}, p)_{111}^{k_1\beta_1\alpha_1} \check{F}(k_1, p, \bar{p}, k_1)_{111}^{k_2\beta_2\alpha_2} \times \cdots \times \check{F}(k_{n-1}, p, \bar{p}, k_{n-1})_{111}^{k_n\beta_n\alpha_n}$$

$$V(\underline{k}, \alpha(\underline{k})) = V^{1p}(\rho) V_{\alpha_1}^{pk_1}(\rho) V_{\alpha_2}^{k_1k_2}(\rho) \ldots V_{\alpha_n}^{k_{n-1}k_n}(\rho),$$

$$V(\underline{k}, \beta(\underline{k}))^* = V_{\beta_n}^{k_{n-1}k_n}(\rho)^* V_{\beta_{n-1}}^{k_{n-2}k_{n-1}}(\rho)^* \cdots V^{1p}(\rho)^*.$$

Hence,

$$\begin{aligned} \hat{F} V_{\omega_n} V_{\omega_n}^* \check{F} = \sum_{\substack{\underline{k}, \alpha(\underline{k}), \beta(\underline{k}), \\ \underline{k}', \alpha(\underline{k}'), \beta(\underline{k}')}} & \hat{F}_{\underline{k}\alpha(\underline{k})\beta(\underline{k})} \check{F}_{\underline{k}'\beta(\underline{k}')\alpha(\underline{k}')} V(\underline{k}, \alpha(\underline{k})) V(\underline{k}, \beta(\underline{k}))^* V_{\omega_n} \\ & \times V_{\omega_n}^* V(\underline{k}', \alpha(\underline{k}')) V(\underline{k}', \beta(\underline{k}'))^* \end{aligned} \tag{6.71}$$

and since

$$\begin{aligned} V(\underline{k}, \beta(\underline{k}))^* V_{\omega_n} &= \delta_{\beta_1\gamma_1}\delta_{k_1 l_1}\delta_{\beta_2\gamma_2}\cdots\delta_{k_n l_n} = \delta_{\underline{k}\,\underline{l}}\delta_{\beta(\underline{k})\gamma}, \\ V^*_{\omega_n} V(\underline{k}', \alpha(\underline{k}')) &= \delta_{\alpha'_1\gamma_1}\delta_{k'_1 l_1}\delta_{\alpha'_2\gamma_2}\cdots\delta_{k'_n l_n} = \delta_{\underline{k}'\,\underline{l}}\delta_{\alpha(\underline{k}')\gamma} \end{aligned} \tag{6.72}$$

we obtain

$$\hat{F} V_{\omega_n} V^*_{\omega_n} \check{F} = \sum_{\substack{\alpha(\underline{l}) \\ \beta(\underline{l})}} \hat{F}_{\underline{l}\alpha(\underline{l})\gamma} \check{F}_{\underline{l}\beta(\underline{l})\gamma} V(\underline{l}, \alpha(\underline{l})) V(\underline{l}, \beta(\underline{l}))^* \tag{6.73}$$

so that

$$\begin{aligned} \operatorname{tr}(\hat{F} V_{\omega_n} V^*_{\omega_n} \check{F}) &= \sum_{\substack{\alpha(\underline{l}) \\ \beta(\underline{l})}} \hat{F}_{\underline{l}\alpha(\underline{l})\gamma} \check{F}_{\underline{l}\beta(\underline{l})\gamma} \delta_{\alpha(\underline{l})\beta(\underline{l})} \\ &= \sum_{\alpha(\underline{l})} \hat{F}_{\underline{l}\alpha(\underline{l})\gamma} \check{F}_{\underline{l}\alpha(\underline{l})\gamma}. \end{aligned} \tag{6.74}$$

Combining Eq. (6.74) with Eq. (5.91) we obtain the following result.

Proposition 6.10.

(i) *The trace vector t^n on $A^p_n \cong \bigoplus_{k \in L} \mathrm{Mat}_{\mu^n_k}(\mathbb{C})$ is given by*

$$t^n = (t^n_k)_{k \in L},$$

$$t^n_k = \begin{cases} \dfrac{d(k)}{d(p)^{n+1}} & \text{if } \mu^n_k \neq 0 \\ 0 & \text{otherwise}. \end{cases} \tag{6.75}$$

(ii) *The Markov trace on A^p_{n+1} extends the Markov trace on A^p_n, for all n.* □

Proof. From Eqs. (5.91) and (6.74) one infers that

$$t^n_{l_n} = \frac{|\lambda_p|^{n+1}}{|\lambda_{l_n}|} = \frac{1}{d(p)^{n+1}} d(l_n) \tag{6.76}$$

and (i) follows.

(ii) Since $d = (d(k))_{k \in L}$ is the Perron-Frobenius eigenvector of $\mathbb{N}_p$ to the eigenvalue $d(p)$, by Lemma 6.3, it follows that,

$$t^n_k = \sum_l t^{n+1}_l N^l_{pk} \tag{6.77}$$

and this implies that the trace on A^p_{n+1} extends the trace on A^p_n. ■

To make the preceding discussion complete, we add the following obvious remarks:

(i) If we consider the multimatrix chain

$$M^{p_0}_0 \subseteq M^{p_0}_1 \subseteq \cdots \tag{6.78}$$

associated to the sectors p_0, p_1, p_2, ... then the $R^{\pm}$-matrices no longer define a representation of the braid groups on a path space. Nevertheless, the operators

$$R^{\pm}(p_n, p_{n+1}) = \sum_{\substack{\omega_{n-1},\\ m,j,k,\alpha,\beta,\\ \gamma,\delta}} R^{\pm}(l, p_n, p_{n+1}, m)^{j\gamma\delta}_{k\alpha\beta}$$

$$\times V_{\omega_{n-1}} V^{lj}_{\gamma}(\rho_{n+1}) V^{jm}_{\delta}(\rho_n) V^{km}_{\beta}(\rho_{n+1})^* V^{lk}_{\alpha}(\rho_n)^* V^*_{\omega_{n-1}} \tag{6.79}$$

define isometries between the path spaces associated to the Bratteli diagrams of p_0, $p_1, \dots p_{n-1}, p_n, p_{n+1}, \dots$ and $p_0, p_1, \dots p_{n-1}, p_{n+1}, p_n, \dots$.

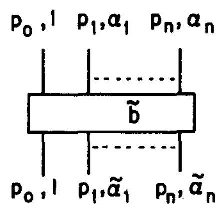
(ii) Just as in the case of a single sector, it is possible to define a faithful trace on the multi-matrix chain associated to the sequence $p_0, p_1, p_2, \dots p_n, \dots$. If we represent the matrix element of $b \in A^{p_0}_n$

$$\langle V_{\omega_n}; bV_{\tilde{\omega}_n} \rangle,$$

$$V_{\omega_n} = V^{1p_0}(\rho_0) V^{p_0 k_1}_{\alpha_1}(\rho_1) V^{k_1 k_2}_{\alpha_2}(\rho_2) \dots V^{k_{n-1} k_n}_{\alpha_n}(\rho_n) \tag{6.80}$$

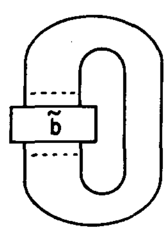
$$V_{\tilde{\omega}_n} = V^{1p_0}(\rho_0) V^{p_0 \tilde{k}_1}_{\tilde{\alpha}_2}(\rho_1) V^{\tilde{k}_1 \tilde{k}_2}_{\tilde{\alpha}_2}(\rho_2) \dots V^{\tilde{k}_{n-1} k_n}_{\tilde{\alpha}_n}(\rho_n)$$

by the diagram



(6.81)

then, using the conventions of Sec. 5,



(6.82)

defines a trace on $A^{p_0}_n$. It is easy to check that this trace is faithful, with trace vector $t^n = (t^n_k)_{k \in I}$,

$$t^n_k = \frac{1}{d(p_0)} \frac{1}{d(p_1)} \cdots \frac{1}{d(p_n)} \cdot \begin{cases} d(k) & \text{if } \mu^n_k \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.83}$$

and that the trace on $A^{p_0}_{n+1}$ extends the trace on $A^{p_0}_n$.

(iii) On the tower (6.37) the vector

$$t^n = \frac{1}{d(p)^{n+1}} \cdot d, \qquad n = 0, 1, 2, \ldots \tag{6.84}$$

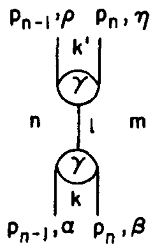
defines a (positive) Markov trace of modulus $\beta = d(p)^2$ in the sense of Goodman, de la Harpe and Jones [51]. Any other (positive) Markov trace on the tower (6.37) has a trace vector which differs from (6.84) only by a multiplicative constant.

(iv) It is easy, albeit lengthy, to check that the trace φ_n defined on $\rho^n(\mathscr{A})'$ coincides with the normal functional $(\phi_\rho)^n$ on $\rho^n(\mathscr{A})'$, where ϕ_ρ is the unique regular left inverse of the irreducible morphism ρ; see [22].

We now return to the study of the path algebra of a collection of sectors p_0, p_1, $\ldots$, p_{n-1}, p_n, $\ldots$ and the decomposition of the product $p_{n-1} \times p_n$ into irreducible subrepresentations:

$$p_{n-1} \times p_n \cong \bigoplus_{l \in L} \bigoplus_{\alpha=1}^{N^l_{p_n p_{n-1}}} l^{(\alpha)}. \tag{6.85}$$

Using the matrix elements $P^{(l,\gamma)}(n, p_{n-1}, p_n, m)^{k'\rho\eta}_{k\alpha\beta}$,



$$\leftrightarrow P^{(l,\gamma)}(n, p_{n-1}, p_n, m)^{k'\rho\eta}_{k\alpha\beta} = \sum_\delta \hat{F}(n, p_{n-1}, p_n, m)^{l\gamma\delta}_{k\alpha\beta} \check{F}(n, p_{n-1}, p_n, m)^{k'\rho\eta}_{l\gamma\delta} \tag{6.86}$$

we define a set of projections in the algebra $A^{p_0}_n$ by

$$P^{(l,\gamma)}(p_{n-1}, p_n) = \sum_{\substack{\omega_{n-2},\\ k_{n-1}, \alpha_{n-1}, \alpha_n,\\ k'_{n-1}, \alpha'_{n-1}, \alpha'_n,\\ k_n}} P^{(l,\gamma)}(k_{n-2}, p_{n-1}, p_n, k_n)^{k'_{n-1}\alpha'_{n-1}\alpha'_n}_{k_{n-1}\alpha_{n-1}\alpha_n} V_{\omega_{n-2}} V^{k_{n-2}k'_{n-1}}_{\alpha'_{n-1}}(\rho_{n-1})$$
$$\times V^{k'_{n-1}k_n}_{\alpha'_n}(\rho_n) V^{k_{n-1}k_n}_{\alpha_n}(\rho_n)^* V^{k_{n-2}k_{n-1}}_{\alpha_{n-1}}(\rho_{n-1})^* V^*_{\omega_{n-2}}. \tag{6.87}$$

Let us denote by $\mathbb{C}\Omega_n$ the path space on which the algebra $A^{p_0}_n$ acts and by $\mathbb{C}\Omega_{n-2,l}$ the path space on which

$$A^{p_0}_{n-2,l} = (\rho_0 \circ \cdots \circ \rho_{n-2} \circ \rho_l(\mathscr{A}))', \qquad \rho_l \cong l \in L. \tag{6.88}$$

acts.

Lemma 6.10. *The set $\{P^{(l,\gamma)}(p_{n-1}, p_n)\}_{l\in L, \gamma=1,\dots N^l_{p_n p_{n-1}}}$ consists of selfadjoint, orthogonal projections belonging to the path algebra $A_n^{p_0}$, which satisfy*

$$\sum_{l,\gamma} P^{(l,\gamma)}(p_{n-1}, p_n) = \mathbb{1}|_{A_n^{p_0}}. \tag{6.89}$$

Each projection $P^{(l,\gamma)}(p_{n-1}, p_n)$ projects $\mathbb{C}\Omega_n$ onto a subspace $V^{(l,\gamma)}$ which is isomorphic to $\mathbb{C}\Omega_{n-2,l}$. □

Proof. From the definition (6.87) of the projections $P^{(l,\gamma)}(p_{n-1}, p_n)$, it follows that self-adjointness is equivalent to

$$\overline{P^{(l,\gamma)}(n,i,j,m)^{k\alpha\beta}_{k'\eta\rho}} = P^{(l,\gamma)}(n,i,j,m)^{k'\eta\rho}_{k\alpha\beta}. \tag{6.90}$$

This latter equation follows immediately from the definition of $P^{(l,\gamma)}(n,i,j,m)^{k\alpha\beta}_{k'\eta\rho}$, Eq. (6.86), and the fact that $\hat{F}$ and $\check{F}$ are unitary matrices, inverse to each other. Orthogonality of the projections and completeness, Eq. (6.89), follow from the corresponding properties of $P^{(l,\gamma)}(n,i,j,m)$, Eqs. (4.101) and (4.102). Finally, we notice that the linear mappings

$$\begin{aligned} \mathscr{U}_{(l,\gamma)}&: \mathbb{C}\Omega_n \to \mathbb{C}\Omega_{n-2,l}, \\ \tilde{\mathscr{U}}_{(l,\gamma)}&: \mathbb{C}\Omega_{n-2,l} \to \mathbb{C}\Omega_n, \end{aligned} \tag{6.91}$$

defined on the basis vectors of the respective path spaces by

$$\begin{aligned} \mathscr{U}_{(l,\gamma)} V_{\omega_{n-2}} V^{k_{n-2}k_{n-1}}_{\alpha_{n-1}}(\rho_{n-1}) V^{k_{n-1}k_n}_{\alpha_n}(\rho_n) &= \sum_{\delta} \hat{F}(k_{n-2}, p_{n-1}, p_n, k_n)^{l\gamma\delta}_{k_{n-1}\alpha_{n-1}\alpha_n} V_{\omega_{n-2}} V^{k_{n-2}k_n}_{\delta}(\rho^l) \\ \tilde{\mathscr{U}}_{(l,\gamma)} V_{\omega_{n-2}} V^{k_{n-2}k_n}_{\delta}(\rho^l) &= \sum_{k'_{n-1},\beta_{n-1},\beta_n} \check{F}(k_{n-2}, p_{n-1}, p_n, k_n)^{k'_{n-1}\beta_{n-1}\beta_n}_{l\gamma\delta} V_{\omega_{n-2}} \\ &\quad \times V^{k_{n-2}k'_{n-1}}_{\beta_{n-1}}(\rho_{n-1}) V^{k'_{n-1}k_n}_{\beta_n}(\rho_n) \end{aligned} \tag{6.92}$$

are partial isometries with the properties

$$\mathscr{U}_{(l,\gamma)} \tilde{\mathscr{U}}_{(l,\gamma)} = \mathbb{1}|_{\mathbb{C}\Omega_{n-2,l}} \tag{6.93}$$

$$\tilde{\mathscr{U}}_{(l,\gamma)} \mathscr{U}_{(l,\gamma)} = P^{(l,\gamma)}(p_{n-1}, p_n). \tag{6.94}$$

This completes the proof of Lemma 6.10. ■

Next, we consider the tower (6.37). Then for all $n \geq 2$ the product $p_{n-1} \times p_n \cong p \times \bar{p}$ contains the vacuum representation 1 precisely once. We introduce the notation

$$e_n = P^{(1,1)}(p_{n-1}, p_n), \qquad n = 2, 3, 4, \dots \tag{6.95}$$

for the projections corresponding to the vacuum sector. By definition, Eq. (6.87),

$$e_n = \sum_{\substack{\omega_{n-2},\\ k_{n-1},\alpha_{n-1},\alpha_n,\\ k'_{n-1},\alpha'_{n-1},\alpha'_n,\\ k_n}} P^{(1,1)}(k_{n-2},p,\bar{p},k_n)^{k'_{n-1}\alpha'_{n-1}\alpha'_n}_{k_{n-1}\alpha_{n-1}\alpha_n} V_{\omega_{n-2}} V^{k_{n-2}k'_{n-1}}_{\alpha'_{n-1}}(\rho)$$

$$\times V^{k'_{n-1}k_n}_{\alpha'_n}(\bar{\rho}) V^{k_{n-1}k_n}_{\alpha_n}(\bar{\rho})^* V^{k_{n-2}k_{n-1}}_{\alpha_{n-1}}(\rho)^* V^*_{\omega_{n-2}}, \tag{6.96}$$

for n even (for n odd, exchange p and $\bar{p}$). The sum over ω_{n-2} in Eq. (6.96) is a sum over all paths of length $n-2$ on the Bratteli diagram (6.25).

Lemma 6.11.

$$P^{(1,1)}(k_{n-2},p_{n-1},p_n,k_n)^{k'_{n-1}\alpha'_{n-1}\alpha'_n}_{k_{n-1}\alpha_{n-1}\alpha_n} = \delta_{k_{n-2}k_n}\delta_{\alpha_{n-1}\alpha_n}\delta_{\alpha'_{n-1}\alpha'_n}\frac{d(k_{n-1})^{1/2}d(k'_{n-1})^{1/2}}{d(p)d(k_{n-2})} \tag{6.97}$$

so that

$$e_n = \sum_{\substack{\omega_{n-2},\\ k_{n-1},k'_{n-1},\\ \alpha_n,\alpha'_n}} \frac{d(k_{n-1})^{1/2}d(k'_{n-1})^{1/2}}{d(p)d(k_{n-2})} V_{\omega_{n-2}} V^{k_{n-2}k'_{n-1}}_{\alpha'_n}(\rho) V^{k'_{n-1}k_{n-2}}_{\alpha'_n}(\bar{\rho})$$

$$\times V^{k_{n-1}k_{n-2}}_{\alpha_n}(\bar{\rho})^* V^{k_{n-2}k_{n-1}}_{\alpha_n}(\rho)^* V^*_{\omega_{n-2}} \tag{6.98}$$

holds, for n even (for n odd, exchange ρ and $\bar{\rho}$). □

Proof. Equation (5.91) and Eq. (6.86) imply (6.97) which in turn implies (6.98). ■

Theorem 6.12. *The projections $\{e_n\}_{n\geq 2}$ on the tower (6.37) satisfy the Temperley-Lieb algebra*

$$\text{(i)} \qquad e_n e_m = e_m e_n,\ |m-n| \geq 2,$$

$$\text{(ii)} \qquad \beta e_n e_{n\pm 1} e_n = e_n,\ \forall n \geq 0, \tag{6.99}$$

and, moreover,

$$\text{(iii)} \qquad \beta\, \mathrm{tr}_M(xe_n) = \mathrm{tr}_M(x)$$

holds, for all $x \in A_{n-1}$ and for $\beta = d(p)^2$.

The projections $\{e_n\}_{n\geq 0}$ are Jones projections, in the sense that $e_n \in A_n$ implements the conditional expectation of A_{n-1} onto A_{n-2}:

$$e_n x e_n = E_{n-1}(x)e_n, \qquad \forall x \in A_{n-1}, \tag{6.100}$$

where $E_{n-1}\colon A_{n-1} \to A_{n-2}$ is the unique conditional expectation of A_{n-1} onto A_{n-2} which is compatible with the Markov trace on A_{n-1}. □

Proof. Using formula (6.75) for the trace vectors t^{n-1}, t^{n-2}, we obtain

$$e_n = \sum_{\substack{\omega_{n-2},\\ k_{n-1}, k'_{n-1},\\ \alpha_n, \alpha'_n}} \frac{(t^{n-1}_{k_{n-1}})^{1/2} (t^{n-1}_{k'_{n-1}})^{1/2}}{t^{n-2}_{k_{n-2}}} V_{\omega_{n-2}} V^{k_{n-2}k'_{n-1}}_{\alpha'_n}(\rho) V^{k'_{n-1}k_{n-2}}_{\alpha'_n}(\bar{\rho})$$

$$\times V^{k_{n-1}k_{n-2}}_{\alpha_n}(\bar{\rho})^* V^{k_{n-2}k_{n-1}}_{\alpha_n}(\rho)^* V^*_{\omega_{n-2}}. \tag{6.101}$$

Equation (6.101) is none other than the well-known formula of Ocneanu and Sunder for the Jones projection e_n [51, 52, 53]. The rest of the theorem follows, for example, from [53, Proposition 6].

Remark 6.13. The Temperley-Lieb relations (6.9) for the projections e_n were first derived in [22], in the context of two-dimensional theories. □

As an illustration of the preceding arguments, we now treat the example of a self-conjugate sector $p \in L$ which satisfies the composition rule

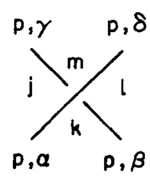
$$p \times p \cong 1 \oplus q, \qquad q \in L; \tag{6.102}$$

see also [22, 31].

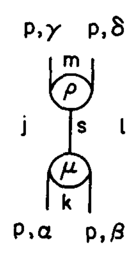
Models which satisfy this composition rule are found in two-dimensional conformal field theory, for example the $SU(2) - WZW$ model [65, 66, 67]. In this case, the representations of the braid groups defined in Lemma 6.4 (i) and Remark 6.6 (ii) are given by C^*-representations of Hecke algebras of type A_n, as analysed in [59]. Combining Remark 6.6 (ii) and Lemma 6.4 (ii), we obtain, for p satisfying Eq. (6.102), representations of the braid groups on the tower characterized by the inclusion matrix $\mathbb{N}_p$. We will show that these representations coincide with the ones defined in [60] using Jones' projections.

The next lemma tells us how to recover an arbitrary braid matrix element from fusion matrix elements and braid matrices acting on the vacuum sector.

Lemma 6.14.



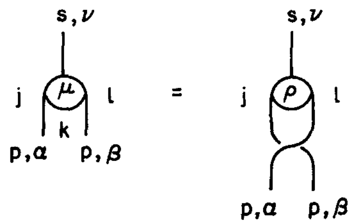
$$= \sum_{s,\rho,\mu} R^+(\bar{s}, p, p, 1)^{\bar{p}\rho 1}_{\bar{p}\mu 1}$$



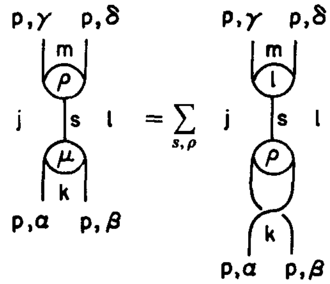
(6.103)

□

Proof. From Eq. (4.122),

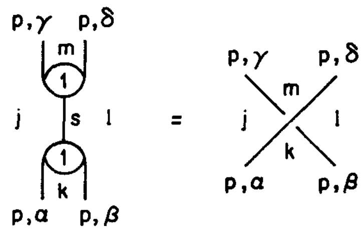
$$\sum_{\mu} R^{+}(\bar{s}, p, p, 1)^{\bar{p}\rho 1}_{\bar{p}\mu 1}$$  (6.104)

it follows that

$$\sum_{s,\rho,\mu} R^{+}(\bar{s}, p, p, 1)^{\bar{p}\rho 1}_{\bar{p}\mu 1}$$  (6.105)

and our assertion follows from (4.99). ■

For a sector satisfying Eq. (6.102), Eq. (6.103) reduces to

$$\sum_{s=1,q} R^{+}(\bar{s}, p, p, 1)^{p11}_{p11}$$  (6.106)

Let us introduce the following notation:

$$\varphi(s) := e^{2\pi i\theta_p(s)} := R^{+}(\bar{s}, p, p, 1)^{\bar{p}11}_{\bar{p}11}, \tag{6.107}$$

$$\varphi(1) := e^{2\pi i\theta_{p,\bar{p}}} = R^{+}(1, p, \bar{p}, 1)^{\bar{p}11}_{p11}, \tag{6.108}$$

so that, using Eq. (6.86), we have that

$$\varphi(1)P^{(1,1)}(j, p, p, l)^{m\gamma\delta}_{k\alpha\beta} + \varphi(q)P^{(q,1)}(j, p, p, l)^{m\gamma\delta}_{k\alpha\beta} = R^{+}(j, p, p, l)^{m\gamma\delta}_{k\alpha\beta}. \tag{6.109}$$

This means that the representation π of the braid groups, generated by the R^{+}-matrices on the tower (6.37), is of the following form:

$$\pi(\sigma_n) = \varphi(1)e_n + \varphi(q)(\mathbb{1} - e_n), \tag{6.110}$$

where we used Eq. (6.89). Rescaling $\pi(\sigma_n)$ we obtain

$$g_n = qe_n + (1 - e_n), \tag{6.111}$$

with

$$g_n = \overline{\varphi(q)}\pi(\sigma_n), \qquad q = \varphi(1)\cdot\overline{\varphi(q)}. \tag{6.112}$$

Proposition 6.15. [31]
(i) *If a sector* $p \in L$ *satisfies Eq.* (6.102), *the representation*

$$\sigma_n \mapsto g_n = qe_n + (1 - e_n) \tag{6.113}$$

on the tower (6.37) *coincides with the representation defined by Jones in* [60].
(ii) *The phase* $q = \varphi(1)\cdot\overline{\varphi(q)}$ *is related to the index* $d(p)^2$ *of the tower by*

$$d(p)^2 = q + q^{-1} + 2. \tag{6.114}$$

(iii) *The statistical dimension* $d(p)$ *of the sector* p *satisfies*

$$d(p) \leq 2. \tag{6.115}$$

□

For the proof, we refer to [31, 60]. The previous considerations allow us to treat a more general case: if a sector $p \in L$ satisfies the composition rule

$$p \times p = j \oplus k, \qquad j, k \in L, \tag{6.116}$$

then the braid group representations on the multi-matrix chain (6.35) (i) are given by

$$\pi(\sigma_n) = \varphi(j)P^{(j,1)}(p_{n-1}, p_n) + \varphi(k)P^{(k,1)}(p_{n-1}, p_n). \tag{6.117}$$

It is easy to check that the elements

$$g_n = -\overline{\varphi(k)}\cdot\pi(\sigma_n) \tag{6.118}$$

satisfy the defining relations of the Hecke algebra $H_n(q)$, with parameter $q = -\varphi(j)\cdot\overline{\varphi(k)}$. The representation theory of these algebras, equipped with a positive Markov trace, has been studied by Wenzl in [59]. It is possible to derive, from the work of Wenzl, restrictions on the possible values of the parameters q and λ_p, (see also [22]). Examples of models which contain superselection sectors satisfying Eq. (6.116) appear to be $SU(n)$ Chern-Simons theories in 2 + 1 dimensions [68, 23, 69, 70].

Recently, Wenzl [64] has given a complete list of unitary representations of Birman-Wenzl algebras admitting a positive, faithful Markov trace. His results allow us to study a three-channel self-conjugate sector $p \in L$, provided it satisfies some additional conditions. Such superselection sectors are expected to arise in $SO(n)$ Chern-Simons models.

Proposition 6.16. *Let $p \in L$ be a three-channel self-conjugate sector, that is,*

$$p \times p \cong 1 \oplus j \oplus k, j, k \in L. \tag{6.119}$$

The elements

$$g_n = i\varphi(j)^{-1/2} \cdot \varphi(k)^{-1/2} \cdot \pi(\sigma_n) \tag{6.120}$$

satisfy the defining relations of a Birman-Wenzl algebra provided the following two conditions hold. If we define

$$x = 1 - \frac{\cos \pi(\theta_p(j) + \theta_p(k) - 2\theta_{p,\bar{p}})}{\cos \pi(\theta_p(k) - \theta_p(j))}$$

then

$$d(p) = |x|, \tag{6.121}$$

and

$$\varphi(k) \cdot \varphi(j) = -\mathrm{sign}(x) \tag{6.122}$$

where $\varphi(s) = e^{2\pi i \theta_p(s)}$, $s = j, k$. □

Like in the case of Hecke algebras, this allows to derive from [64] restrictions on the possible values of the parameters $\theta_{p,\bar{p}}$, $\theta_p(j)$ and $\theta_p(k)$.

Proof. For p satisfying (6.119) one finds

$$\pi(\sigma_n) = \varphi(1)P^{(1,1)}(p_{n-1}, p_n) + \varphi(j)P^{(j,1)}(p_{n-1}, p_n) + \varphi(k)P^{(k,1)}(p_{n-1}, p_n). \tag{6.123}$$

The defining relations of a Birman-Wenzl algebra are [64]
(i) $g_n g_{n\pm1} g_n = g_{n\pm1} g_n g_{n\pm1}$
(ii) $g_n g_m = g_m g_n$, $|n - m| \geq 2$
(iii) $(g_n - r^{-1})(g_n + q^{-1})(g_n - q) = 0$, $q, r \in \mathbb{C}$
(iv) $e_n g_{n-1}^{\pm1} e_n = r^{\pm1} e_n$
for invertible g_n's. In (iv), the element e_n is defined by

$$(q - q^{-1})(1 - e_n) = g_n - g_n^{-1}. \tag{6.124}$$

Equations (i) and (ii) are the defining relations of the braid group and hence are satisfied by $\pi(\sigma_n)$. If one rescales $\pi(\sigma_n)$, setting

$$g_n = \alpha \cdot \pi(\sigma_n), \qquad \alpha^2 = -\varphi(j)^{-1}\varphi(k)^{-1} \tag{6.125}$$

it is easy to check that the g_n's also satisfy (iii), with

$$q = \alpha \cdot \varphi(k) = -\alpha^{-1} \cdot \varphi(j)^{-1} \tag{6.126}$$

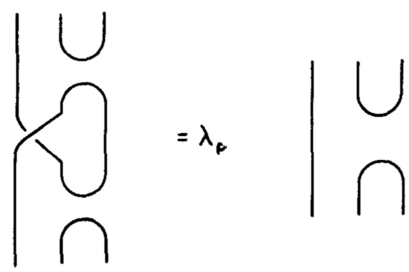
and

$$r = \alpha^{-1} \varphi(1)^{-1}. \tag{6.127}$$

If we define e_n by Eq. (6.124), it follows from (iii), (6.123) and (6.89) that

$$e_n = x P^{(1,1)}(p_{n-1}, p_n), \qquad x = \frac{q - q^{-1} + r - r^{-1}}{q - q^{-1}}. \tag{6.128}$$

Furthermore, the graphical equation



$$= \lambda_p \tag{6.129}$$

which follows from Lemma 5.1, implies that

$$P^{(1,1)}(p_{n-1}, p_n) \pi(\sigma_{n-1}) P^{(1,1)}(p_{n-1}, p_n) = \lambda_p P^{(1,1)}(p_{n-1}, p_n) \tag{6.130}$$

holds, where $\lambda_p = \frac{1}{d(p)} \overline{\varphi(1)}$, by (6.27). It then follows from (6.125) and (6.128) that (iv) is satisfied if and only if

$$x \alpha \lambda_p = r. \tag{6.131}$$

Taking absolute values on both sides we see that

$$d(p) = |x|, \qquad x = 1 + \frac{r - r^{-1}}{q - q^{-1}} \in \mathbb{R} \tag{6.132}$$

and comparing phases, we obtain

$$\overline{\varphi(1)}\, \mathrm{sign}(x) = r\bar{\alpha}. \tag{6.133}$$

Combining (6.133) and (6.127) with (6.125) one finds (6.122), while (6.121) is obtained by expressing the parameters r and q in (6.132) in terms of the physical quantities $\varphi(1)$, $\varphi(j)$ and $\varphi(k)$. ■

The following Corollary is an immediate consequence of the results in [64].

Corollary 6.17.

(a) *The pair* $(r, q) \in U(1) \times U(1)$ *defined by Eqs.* (6.126) *and* (6.127) *can take only the values*

$$q = e^{\pi i/l} \qquad \text{and} \qquad r = q^n \tag{6.134}$$

with l *and* n *satisfying one of the following six conditions*:

(i) $n = 0$ *and* l *is arbitrary.*
(ii) $n = 1$ *and* l *is arbitrary.*
(iii) $n = 2$ *and* $l \in \mathbb{N}/2$.
(iv) $2 < n < l - 2$ *and* $l \in \mathbb{N}$.
(v) $-l < n < -1$, n even and $l \in \mathbb{N}$, l *odd.*
(vi) $-l < n < -1$, n *odd and* $l \in \mathbb{N}$, l *even.*

Moreover, (r, q) *can take all values obtained from* (6.134) (i)–(vi) *by one of the following transformations*:

$$(r, q) \to (-r, -q)$$

$$(r, q) \to (r^{-1}, q^{-1})$$

$$(r, q) \to (r, -q^{-1}).$$

To each choice of (r, q) *in the classes* (i) *to* (vi), *there corresponds an associated principal graph* $\Gamma(r, q)$ *which completely characterizes the corresponding unitary representation of the braid group* B_∞.

(b) *For* r *and* q *chosen as in* (6.134), (i)–(vi) *this implies that*

$$\theta_{p,\bar{p}} = \frac{-n}{2l}\left(\operatorname{mod}\frac{1}{4}\mathbb{Z}\right) \tag{6.135}$$

$$\theta_p(k) = \frac{1}{2l}\left(\operatorname{mod}\frac{1}{4}\mathbb{Z}\right) \tag{6.136}$$

$$\theta_p(j) = \frac{-1}{2l}\left(\operatorname{mod}\frac{1}{4}\mathbb{Z}\right). \tag{6.137}$$

□

Proof. (a) is merely a restatement of some of the results obtained in [64] for unitary representations of Birman-Wenzl algebras carrying a positive Markov trace. For more details we refer to [64].

(b) Since by Eqs. (6.125)–(6.127)

$$\begin{aligned} q &= e^{\pi i(\theta_p(k) - \theta_p(j) + 1/2)} \\ r &= e^{\pi i(\theta_p(j) + \theta_p(k) - 2\theta_{p,\bar{p}} - 1/2)} \end{aligned} \tag{6.138}$$

it follows from $q = e^{\pi i/l}$ that

$$\theta_p(k) - \theta_p(j) = \frac{1}{l}\left(\mathrm{mod}\,\frac{1}{2}\mathbb{Z}\right). \tag{6.139}$$

Furthermore, the constraint (6.122) implies that

$$\theta_p(k) + \theta_p(j) = 0\left(\mathrm{mod}\,\frac{1}{2}\mathbb{Z}\right). \tag{6.140}$$

From (6.139) and (6.140) one easily derives (6.136) and (6.137). Equations (6.138), (6.140) together with $r = q^n$ mean that

$$2\theta_{p,\bar{p}} = \frac{-n}{l}\left(\mathrm{mod}\,\frac{1}{2}\mathbb{Z}\right)$$

or

$$\theta_{p,\bar{p}} = \frac{-n}{2l}\left(\mathrm{mod}\,\frac{1}{4}\mathbb{Z}\right).$$

This completes the proof of the corollary. ∎

Finally, we remark that one expects to extract further information on the structure of the towers (6.37) by using methods developed by Ocneanu [61].

6.2. *Projective representations of mapping class groups*

Next, we show that starting from the statistics and fusion data of a three dimensional algebraic field theory with braid statistics, one can define matrices S and T which generate a projective representation of $SL(2, \mathbb{Z})$.

Following ideas in conformal field theories [63], we define an $|L| \times |L|$ matrix $\psi = (\psi_{ij})$, $i, j \in L$, by

$$\psi_{ij} := d(i)d(j) \tag{6.141}$$

$$= d(i)d(j) \sum_{k,\alpha,\beta} R^+(i, \bar{i}, \bar{j}, \bar{j})^{k\alpha\beta}_{111} \cdot R^+(i, \bar{j}, \bar{i}, \bar{j})^{111}_{k\alpha\beta}\,.$$

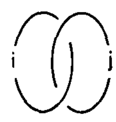
It is easy to see that the matrix elements ψ_{ij} satisfy the following properties.

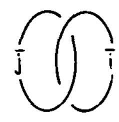
Lemma 6.18.
(i) $\psi_{i1} = \psi_{1i} = d(i)$
(ii) $\psi_{ij} = \psi_{\bar{j}\bar{i}} = \bar{\psi}_{i\bar{j}} = \psi_{ji}$

(iii) $\psi_{ij} = \sum_k N_{ij}^k e^{2\pi i(s_i+s_j-s_k)} d(k) = \sum_k N_{ij}^k e^{2\pi i(\theta_{i,\bar{i}}+\theta_{j,\bar{j}}-\theta_{k,\bar{k}})} d(k)$

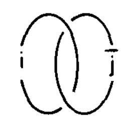
(iv) $\frac{1}{d(j)} \psi_{kj}\psi_{ij} = \sum_m N_{ik}^m \psi_{mj}$. □

Proof. (i) is obvious. To prove (ii), notice that by using Reidemeister moves of type II and III, one has that



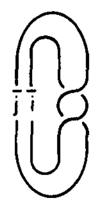
=


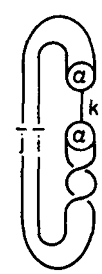
(6.142)

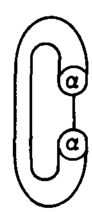
=


(6.143)

Equation (6.142) implies that $\psi_{ij} = \psi_{\bar{j}\bar{i}}$, and (6.143) shows that $\psi_{ij} = \overline{\psi_{i\bar{j}}}$, since $d(s) = d(\bar{s})$ holds for all $s \in L$, (Corollary 5.9 (ii)). To prove (iii), one uses Eq. (4.99), Theorem 4.8 and Lemma 5.13,

$\psi_{ij} = \psi_{\bar{i}\bar{j}} =$


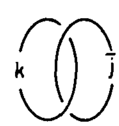
$= d(i)d(j) \sum_{k,\alpha}$


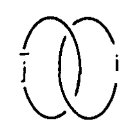
$= d(i)d(j) \sum_{k,\alpha} e^{2\pi i(s_i+s_j-s_k)}$


$$= d(i)d(j) \sum_k e^{2\pi i(s_i+s_j-s_k)} N_{ij}^k \frac{d(k)}{d(i)d(j)}$$

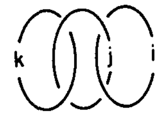
$$= \sum_k e^{2\pi i(s_i+s_j-s_k)} N_{ij}^k d(k).$$

The second equality in (iii) follows from Lemma 5.13 (ii). Equation (iv) follows from

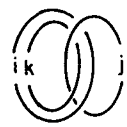
$\frac{1}{d(j)} \psi_{kj}\psi_{ij} = d(k)d(j)d(i)$




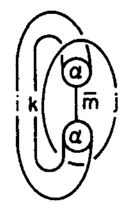
$$= d(k)d(j)d(i)$$



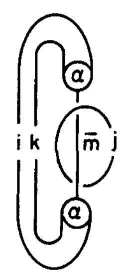
$$= d(k)d(j)d(i)$$



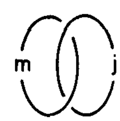
$$= \sum_{m,\alpha} d(k)d(i)d(j)$$



$$= \sum_{m,\alpha} d(k)d(j)d(i)$$



$$= \sum_{m} N_{ik}^{m} d(m)$$



$$= \sum_{m} N_{ik}^{m} \psi_{mj}, \tag{6.144}$$

where Eq. (6.144) follows from Lemma 5.13(i). This completes the proof of Lemma 6.18. ■

Point (iv) of Lemma 6.18 means that the vectors

$$\varphi_j = (\psi_{mj})_{m \in L} \tag{6.145}$$

are common eigenvectors of the fusion rules $\mathbb{N}_i$,

$$\mathbb{N}_i \varphi_j = \lambda_{ij} \varphi_j, \; i, j \in L$$

with eigenvalues $\lambda_{ij} := \dfrac{\psi_{ij}}{d(j)}$. Now the point is that if the matrix ψ is invertible, then it diagonalizes the fusion rules,

$$N_{ik}^{m} = \sum_{j} \frac{\psi_{kj}\psi_{ij}\psi_{jm}^{-1}}{\psi_{1j}}. \tag{6.146}$$

An explicit example of this diagonalization process is given in [43] where the $\hat{su}(2)_{n-1}$ fusion rules are derived. In [62] a criterion is given which settles when ψ is invertible.

Lemma 6.19. [62]

(i) *With respect to the canonical scalar product of* $\mathbb{C}^{|L|}$, *the vectors* φ_j *are either orthogonal,*

$$\langle \varphi_j, \varphi_i \rangle = 0, \qquad i \neq j,$$

or parallel,

$$d(i)\varphi_j = d(j)\varphi_i.$$

(ii) *The matrix* ψ *is invertible if and only if no vector* φ_j *is parallel to* $\varphi_1, j \neq 1$.

(iii) *If* φ_j *is parallel to* $\varphi_1, j \neq 1$, *then the superselection sector* j *obeys permutation statistics.* □

Proof. (i) By Lemma 6.18 (iv) we have that

$$\begin{aligned} \frac{1}{d(j)}\psi_{ij}\langle \varphi_l; \varphi_j \rangle &= \langle \varphi_l; \mathbb{N}_i \varphi_j \rangle \\ &= \sum_{m,k} \overline{\psi_{kl}} N_{ik}^{m} \psi_{mj} \\ &= \sum_{m,k} (\overline{N_{\bar{i}m}^{\bar{k}} \psi_{\bar{k}\bar{l}}}) \psi_{mj} \\ &= \sum_{m} \frac{1}{d(\bar{l})} \psi_{i\bar{l}} \psi_{\overline{m}i} \psi_{mj} \\ &= \frac{1}{d(l)} \psi_{il} \langle \varphi_l; \varphi_j \rangle. \end{aligned}$$

Hence, if $\langle \varphi_l; \varphi_j \rangle \neq 0$ it follows that

$$d(l)\varphi_j = d(j)\varphi_l.$$

(ii) Using again Lemma 6.18 (iv) one finds that

$$\begin{aligned} \langle \varphi_{\bar{i}}; \varphi_k \rangle &= \sum_{j} \overline{\psi_{\bar{i}j}} \psi_{kj} = \sum_{j} \psi_{ij} \psi_{kj} \\ &= \sum_{m,j} N_{ik}^{m} \psi_{mj} d(j) \\ &= \sum_{m} N_{ik}^{m} \langle \varphi_1; \varphi_m \rangle \end{aligned}$$

and, since by assumption,

$$\langle \varphi_1 ; \varphi_m \rangle = \delta_{1m}(\sum d(j)^2)$$

we obtain that

$$\langle \varphi_{\bar{i}} ; \varphi_k \rangle = (\sum d(j)^2)\delta_{\bar{i}k} .$$

This means that

$$\psi^* \psi = (\sum d(j)^2)\mathbb{1} . \tag{6.147}$$

(iii) It follows from Lemma 6.3 and Lemma 6.18 (iii) that $\psi_{ij} = d(i)d(j)$ holds only if $e^{2\pi i(s_i + s_j - s_k)} = 1$, for $N_{ij}^k \neq 0$. The conclusion now follows from Theorem 4.11. ∎

Under the assumptions of the previous lemma, it is possible to rescale ψ so that

$$S = (\sum d(l)^2)^{-1/2}\psi$$

is a unitary matrix, by Eq. (6.147). Next, we define a $|L| \times |L|$ matrix $T = (T_{ij})$ by

$$T_{ij} = e^{-2\pi i c/24} e^{2\pi i s_j} \delta_{ij} . \tag{6.148}$$

If the constant c is chosen appropriately, the matrices S and T have the following properties.

Proposition 6.20. *Let* $\sigma = \sum d(i)^2 e^{-2\pi i s_i}$ *and* $e^{-2\pi i c/24} = \left(\frac{\sigma}{|\sigma|}\right)^{1/3}$, *then the matrices* S, T *and the charge conjugation matrix* $C = (\delta_{\bar{i}j})$ *satisfy the following relations*:

$$S^*S = T^*T = \mathbb{1} \tag{6.149}$$

$$TSTSTS = C , \qquad TC = CT = T . \tag{6.150}$$

The constant c *is determined* mod 8. □

Remark 6.21. Equations (6.149), (6.150) imply that S and T generate a projective representation of the modular transformations $\tau \to -(1/\tau)$ and $\tau \to \tau + 1$, just as in two-dimensional conformal field theory. In that context, the constant c is the central charge of the Virasoro algebra. The remark that the modular S matrix diagonalizes the fusion rules was first made by E. Verlinde for rational conformal field theories [63].

Proof. By Lemma 6.18 (iii),

$$\sum_j \psi_{ij} d(j) e^{-2\pi i s_j} = \sum_{k,j} N_{ij}^k e^{2\pi i(s_i - s_j)} d(k) d(j)$$

$$= \sum_k d(k) e^{2\pi i(s_i - s_k)} \left(\sum_j N_{ik}^{\bar{j}} d(\bar{j}) \right)$$

$$= \left(\sum_k d(k)^2 e^{-2\pi i s_k} \right) d(i) e^{2\pi i s_i}$$

$$= \sigma d_i e^{2\pi i s_i}.$$

This last equation implies that

$$\left(\sum_l \overline{\psi_{il}} d(l) e^{2\pi i s_l} \right) \left(\sum_j \psi_{ij} d(j) e^{-2\pi i s_j} \right) = |\sigma|^2 d(i)$$

and summing over i, we obtain

$$\sum_{j,l} d(j) d(l) e^{2\pi i(s_l - s_j)} \sum_i \overline{\psi_{il}} \psi_{ij} = |\sigma|^2 \left(\sum d(i)^2 \right)$$

or, by (6.147),

$$\left(\sum_l d(l)^2 \right) = |\sigma|^2.$$

This latter equation and

$$\sum_j \psi_{ij} e^{-2\pi i s_j} \psi_{kj} = \sum_{j,m} d(j) e^{-2\pi i s_j} N_{ik}^m \psi_{mj}$$

$$= \sigma e^{2\pi i(s_i + s_k)} \overline{\psi_{ik}}$$

are sufficient to derive (6.149), (6.150); see also [62]. ■

The appearance of matrices S and T satisfying the relations of the modular group generators, in the context of three-dimensional, generally massive, local quantum field theories, might seem, a priori, surprising. However, this fact can be understood by noticing that every three-dimensional field theory with braid statistics appears to determine the chiral sector of some two-dimensional conformal field theory on the circle. Heuristically, the argument to see this goes as follows. Let $V^{1i_1}(\rho^{i_1})$, $V_{\alpha_1}^{i_1 i_2}(\rho^{j_1})$, $\ldots$, $V^{i_n 1}(\rho^{j_n})$ be a set of intertwiners between superselection sectors of a $(2+1)$-dimensional quantum field theory, where the morphisms $\rho^{i_1}, \rho^{j_1}, \ldots, \rho^{j_n}$ are localized in spacelike separated cones $\mathscr{C}_0, \mathscr{C}_1, \ldots, \mathscr{C}_n$ with apices $a_0, a_1, \ldots, a_n$ in the $\{t = 0\}$-plane and asymptotic directions $\theta_0, \theta_1, \ldots, \theta_n$. Let us define the vacuum expectation values of these fields

$$B_{i_1, \ldots, i_n}^{j_1, \ldots, j_n}(\underline{\alpha}; \underline{a}; \underline{\theta}) = \langle \Omega; V^{1i_1}(\rho^{i_1}) V_{\alpha_1}^{i_1 i_2}(\rho^{j_1}) \ldots V^{i_n 1}(\rho^{j_n}) \Omega \rangle \tag{6.151}$$

where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$, $\theta = (\theta_0, \theta_1, \ldots, \theta_n)$ and $\underline{a} = (a_0, a_1, \ldots, a_n)$. It is then possible to write the function $B_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}$ in terms of two functions $F_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}$ and $G_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}$,

$$B_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}(\underline{\alpha}; \underline{a}; \underline{\theta}) = F_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}(\underline{\alpha}; \underline{\theta}) G_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}(\underline{\alpha}; \underline{a}; \underline{\theta}) \tag{6.152}$$

where G is single-valued, and F only depends on $\underline{\theta}$, but not on $\underline{a}$. The monodromy properties of F are those of the conformal blocks of a two-dimensional theory. The angles $\theta_0, \theta_1, \ldots, \theta_n$ can be interpreted as the compactified light-cone variables of the chiral sector of a two-dimensional conformal field theory. The functions $F_{i_1,\ldots,i_n}^{j_1,\ldots,j_n}$ can be constructed from the B's by an appropriate scaling limit. Such ideas have previously been discussed in the example of $SU(n)$ topological Chern-Simons theories in [69, 70]. In conclusion, one can argue quite convincingly that general three-dimensional Chern-Simons gauge theories on a space-time manifold $D_R \times \mathbb{R}$, where D_R is a two-dimensional disk of radius R, uniquely determine two-dimensional, chiral conformal field theories on the cylinder $\partial D_R \times \mathbb{R}$, in an appropriate scaling limit ($R \to \infty$).

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